Precalculus

Lesson 14.5: The Area Problem: The integral

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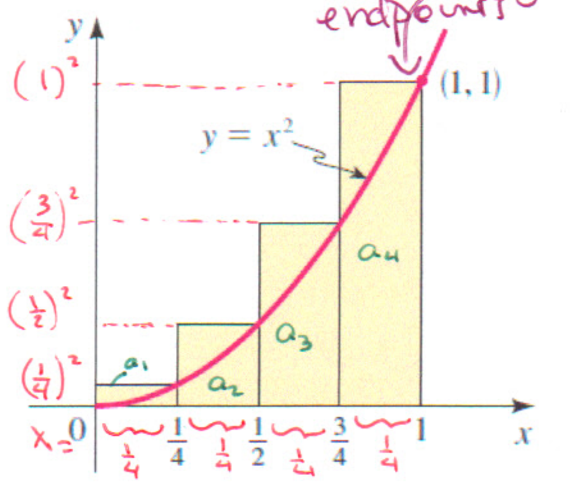
In geometry we found area of polygons. We had set formulas such as the area of a rectangle is length times width. A triangular area is found by calculating ½ the length of the base times the height, and so on. Calculus is used to deal with area problems that have regions containing curved boundaries. Here we can go back to our simple formula for the area of a rectangle and use it to estimate the area of a region under a curve.

Estimating an Area Using Rectangles

Use rectangles to estimate the area under the parabola from 0 to 1.

Using right endpoints



area = (width)(height)

$a_1 = \frac{1}{4}(\frac{1}{4})^2$

$a_2 = \frac{1}{4}(\frac{1}{2})^2$

$a_3 = \frac{1}{4}(\frac{3}{4})^2$

$a_4 = \frac{1}{4}(1)^2$

Add for total area:

$a = a_1 + a_2 + a_3 + a_4$

$= \frac{1}{4}(\frac{1}{4})^2 + \frac{1}{4}(\frac{1}{2})^2 + \frac{1}{4}(\frac{3}{4})^2 + \frac{1}{4}(1^2)$

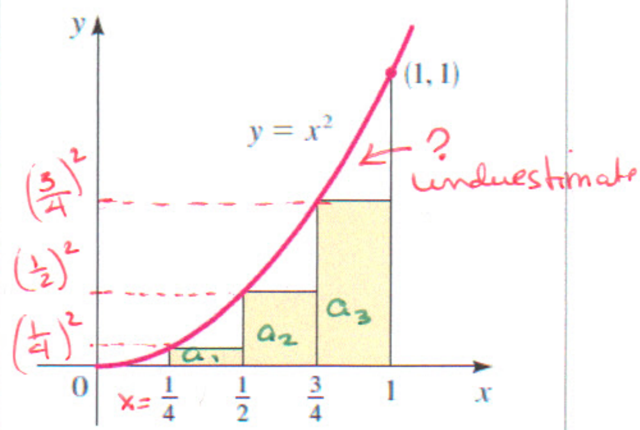
factor out $\frac{1}{4}$

$= \frac{1}{4}\left((\frac{1}{4})^2 + (\frac{1}{2})^2 + (\frac{3}{4})^2 + (1)^2\right)$

do math

$= .46875 \Rightarrow$ over estimate as includes area above curve

Using Left endpoints



now w/ left endpoints 3 rectangles.

$a_1 = \frac{1}{4}(\frac{1}{4})^2$

$a_2 = \frac{1}{4}(\frac{1}{2})^2$

$a_3 = \frac{1}{4}(\frac{3}{4})^2$

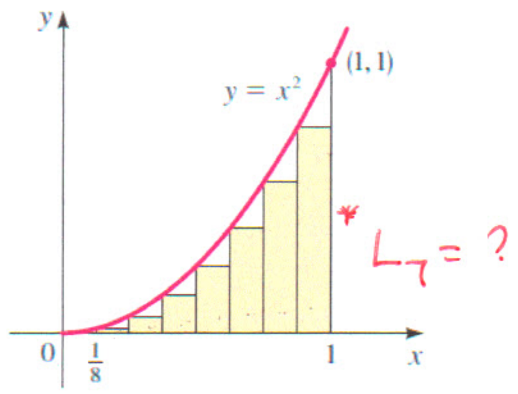
$a = \frac{1}{4}\left((\frac{1}{4})^2 + (\frac{1}{2})^2 + (\frac{3}{4})^2\right)$

$= .21875$

So actual area under curve is between these 2 values

$\boxed{.21875 < a < .46875}$

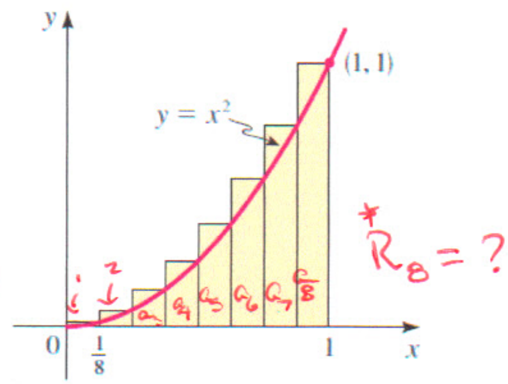
Same problem….smaller rectangles



(a) Using left endpoints

Underestimate of area.

* notation L_n = area using n left endpoints



(b) Using right endpoints

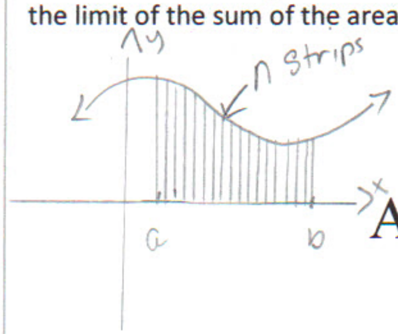
Overestimate of area. * notation:

R_n = area using n right endpoints

The smaller the rectangular strips the more accurate the calculation of the area. This then opens up the door to take a limit as the number of rectangles goes to infinity.

Definition of Area

The area A of the region S that lies under the graph of a continuous function f is the limit of the sum of the areas of the approximating rectangles: ***use right endpoints***.



AREA=(height) x (width)

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

Δx **is the width** of an approximating rectangle,
x_k **is the right endpoint** of the kth **rectangle**
$f(x_k)$ **is its height**.

$\boldsymbol{n}$ *rectangles*
region from $x = a$ *to* $x = b$

width: $\Delta x = \frac{b-a}{n}$

right endpoint: $x_k = a + k\Delta x$

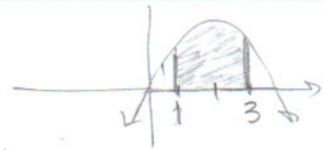
height: $f(x_k) = f(a + k\Delta x)$

Finding an Area under a Curve

Find the area of the region that lies under

$y = 4x - x^2 \quad 1 \le x \le 3$

$\qquad\qquad\quad a \quad\; b$



$\Delta x = \frac{3-1}{n} = \frac{2}{n} = \Delta x \qquad x_k = a + k\Delta x$

$\qquad\qquad\qquad\qquad\qquad\; = 1 + k\left(\frac{2}{n}\right) = 1 + \frac{2k}{n} = x_k$

$f(x_k) = 4\left(1+\frac{2k}{n}\right) - \left(1+\frac{2k}{n}\right)^2$

$-\left(1+\frac{2k}{n}\right)\left(1+\frac{2k}{n}\right)$

$-\left(1+\frac{4k}{n}+\frac{4k^2}{n^2}\right)$

$-1-\frac{4k}{n}-\frac{4k^2}{n^2}$

$= 4 + \frac{8k}{n} - 1 - \frac{4k}{n} - \frac{4k^2}{n^2}$

$f(x_k) = 3 + \frac{4k}{n} - \frac{4k^2}{n^2}$

$A = \lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{4k}{n}-\frac{4k^2}{n^2}\right)\left(\frac{2}{n}\right)$

Separate Σ, pull out constant $\frac{2}{n}$

$= \lim_{n\to\infty} \frac{2}{n}\left(\sum_{k=1}^{n} 3 + \sum_{k=1}^{n} \frac{4k}{n} - \sum_{k=1}^{n} \frac{4k^2}{n^2}\right)$

distribute $\frac{2}{n}$ to each Σ & pull out constants

$= \lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n} 3 + \frac{2}{n}\frac{4}{n}\sum_{k=1}^{n} k - \frac{2}{n}\cdot\frac{4}{n^2}\sum_{k=1}^{n} k^2$

use formulas

$= \lim_{n\to\infty} \cancel{\frac{2}{n}(3n)} + \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$

use associative property

$\lim_{n\to\infty} 6 + \frac{8}{2}\cdot\frac{n}{n}\cdot\frac{n+1}{n} - \frac{8}{6}\cdot\frac{n}{n}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}$

$\lim_{n\to\infty} 6 + 4\left(\frac{n}{n}+\frac{1}{n}\right) - \frac{4}{3}\cdot\left(\frac{n}{n}+\frac{1}{n}\right)\left(\frac{2n}{n}+\frac{1}{n}\right)$ apply limit

$= 6 + 4 - \frac{4}{3}(2)$

$= 10 - \frac{8}{3} = \frac{30}{3} - \frac{8}{3} = \boxed{\frac{22}{3}}$