Precalculus
Lesson 12.5: The Binomial Theorem
Mrs. Snow, Instructor

An expression with two terms is called a **binomial** for example $a + b$ is a binomial. It is an easy enough process to square this binomial or to cube it, but expanding this binomial by a higher degree or multiplying it out more times, will quickly get tedious. Looking at the binomial expansion of $a + b$ for the first five degrees we should see a pattern:

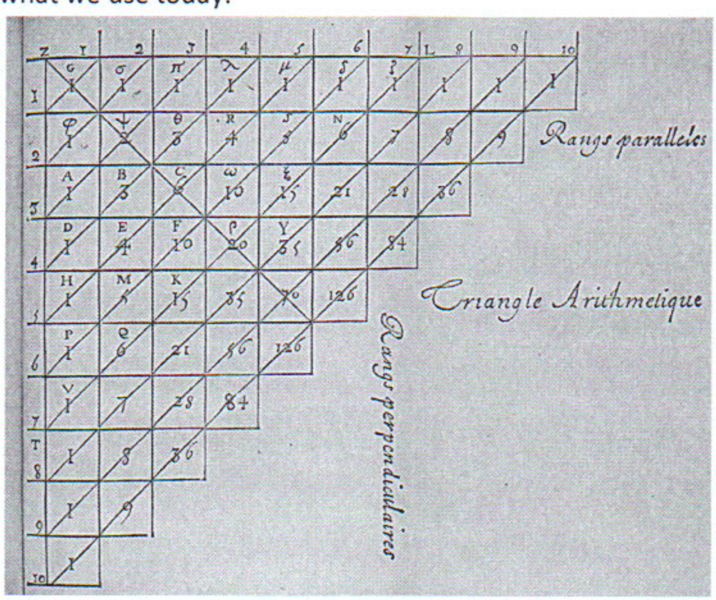
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \quad \text{— 5 terms}$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \quad \text{— 5+1 terms}$$
$$a^5b^0 \quad a^4b^1 \quad a^3b^2 \quad a^2b^3 \quad ab^4 \quad a^0b^5$$

What is the pattern?

$$(a+b)^n$$

1. There are $n + 1$ terms, the first being a^n and the last is b^n.
2. The exponents of a decrease by 1 from term to term while the exponents of b increase by one
3. The sum of the exponents of a and b in each term is n

The pattern that is present in binomial expansion has been known for centuries. Blaise Pascal organized it into a triangular format that has become known as Pascal's Triangle. Below are both his original version and what we use today:



$(a+b)^0$	1
$(a+b)^1$	1 1
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1
$(a+b)^5$	1 5 10 10 5 1
$(a+b)^6$	1 6 15 20 15 6 1
$(a+b)^7$	1 7 21 35 35 21 7 1
$(a+b)^8$	1 8 28 56 70 56 28 8 1
$(a+b)^9$	1 9 36 84 126 126 84 36 9 1
$(a+b)^{10}$	1 10 45 120 210 252 210 120 45 10 1

Using Pascal's Triangle to expand binomials

Expand $(a+b)^7$ coefficients 1 - 7 - 21 - 35 - 35 - 21 - 7 - 1

$= a^7 + 7a^6b^1 + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

$(2-3x)^5$ $a = 2$ $b = -3x$ coefficients: 1 - 5 - 10 - 10 - 5 - 1

$a^5 \quad a^4b \quad a^3b^2 \quad a^2b^3 \quad ab^4 \quad b^5$

$(1)2^5 + (5)2^4(-3x) + (10)2^3(-3x)^2 + (10)2^2(-3x)^3 + (5)2(-3x)^4 + (1)(-3x)^5$

$= 32 - 240x + 720x^2 - 960x^3 + 810x^4 - 243$

Pascal's Triangle is pretty slick for binomial expansions with relatively small values of n. For very large exponents, we need a more efficient way to calculate the coefficients. Pascal's Triangle is recursive in that to find the 100th row, we need the 99th row. So to come up with a process , we will need to use **factorials** that we studied in 12.1.

Binomial Coefficients

If j and n are integers with $0 \le j \le n$, the symbol $\binom{n}{j}$ is defined as

$$\boxed{\binom{n}{j} = \frac{n!}{j!\,(n-j)!}}$$

Calculate the binomial coefficients

$$\binom{9}{4} = \frac{9!}{4!(9-4)!} = \frac{9!}{4!\,5!} = \frac{\overset{3}{\cancel{9}}\cdot\cancel{8}\cdot 7\cdot 6\cdot\cancel{5!}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1\cdot\cancel{5!}} = 3\cdot 7\cdot 6$$

$$= 126$$

$$\binom{100}{3} = \frac{100!}{3!\,97!} = \frac{\overset{50}{\cancel{100}}\cdot\overset{33}{\cancel{99}}\cdot 98\cdot\cancel{97!}}{\cancel{3}\cdot\cancel{2}\cdot 1\cdot\cancel{97!}}$$

$$= 161700$$

This helps up because the values of Pascal's Triangle are in fact binomial coefficients!

n = exponent

$\binom{0}{0}$ $(a+b)^0$

* j = order of term minus 1

$\binom{1}{0}$ $\binom{1}{1}$ $(a+b)^1$

$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$ $(a+b)^2$

$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$ $(a+b)^3$

$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$ $(a+b)^4$

$\binom{5}{0}$ $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$ $(a+b)^5$

* term# . 1 . 2 . 3 . 4 . 5. 6

$\binom{n}{0}$ $\binom{n}{1}$ $\binom{n}{2}$ . . . $\binom{n}{n-1}$ $\binom{n}{n}$

Binomial Theorem

Binomial Theorem

Let x and a be real numbers. For any positive integer n, we have

$$(x+a)^n = \binom{n}{0}x^n + \binom{n}{1}ax^{n-1} + \cdots + \binom{n}{j}a^j x^{n-j} + \cdots + \binom{n}{n}a^n$$
$$= \sum_{j=0}^{n}\binom{n}{j}x^{n-j}a^j \qquad \textbf{(2)}$$

Use the Binomial Theorem to expand the following:

$(x+y)^4$

$$\binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$$\binom{4}{0} = \frac{4!}{0!\,4!} = 1 \qquad \binom{4}{1} = \frac{4!}{1!\,3!} = \frac{4\cdot\cancel{3!}}{\cancel{3!}} = 4 \qquad \binom{4}{2} = \frac{4!}{2!\,2!} = \frac{\overset{2}{\cancel{4}}\cdot 3\cdot\cancel{2!}}{\cancel{2}\cdot\cancel{2!}} = 6$$

$$\binom{4}{3} = \frac{4!}{3!\,1!} = \frac{4\cdot\cancel{3!}}{\cancel{3!}} = 4 \qquad \binom{4}{4} = \frac{\cancel{4!}}{\cancel{4!}} = 1$$

$$\Rightarrow x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$(2y-3)^4$ $\quad a = 2y \quad b = -3 \quad$:) same binomial coefficients above!

$$1(2y)^4 + 4(2y)^3(-3) + 6(2y)^2(-3)^2 + 4\,2y(-3)^3 + 1(-3)^4$$

$$= 16y^4 - 96y^3 + 216y^2 - 216y + 81$$

The Binomial theorem may be used to find a particular term of a binomial expansion:

Based on the expansion of $(x+a)^n$, the term containing x^j is

$$\binom{n}{n-j} a^{n-j} x^j \qquad (3)$$

Find the find the coefficient of y^8 in the expansion of $(2y+3)^{10}$: $n=10,\ j=8$

$x = 2y \quad a = 3$

$$\binom{10}{10-8} = \binom{10}{2} = \frac{10!}{2!\,8!} = \frac{\cancel{10}^{5} \cdot 9 \cdot \cancel{8!}}{\cancel{2} \cdot \cancel{8!}} = 45$$

$$a^{n-j} = 3^{10-8} = 3^2 = 9$$

$$x^j = (2y)^8 = 256y^8$$

$$\text{term} = (45)(9)(256)y^8$$

$$= 103680y^8$$

Find the 6th term in the expansion of $(x+2)^9$

Binomial coefficients: $\binom{9}{5}$ ← exponent ← x^j term is one less than order of term

You forget this fact ?? DO NOT PANIC!

1st	2nd	3rd	4th	5th	6th	
$\binom{9}{0}$	$\binom{9}{1}$	$\binom{9}{2}$	$\binom{9}{3}$	$\binom{9}{4}$	$\binom{9}{5}$	:)
x^9	x^8	x^7	x^6	x^5	x^4 ← and	

$$\binom{9}{5} x^4 (2)^5 = 126x^4(32 = 4032x^4$$

$$\binom{9}{5} = \frac{9!}{5!\,4!} = \frac{\cancel{9}^{3} \cdot \cancel{8} \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{5!}} = 126$$