Precalculus
Lesson 12.3 Geometric Sequences
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Another type of sequence is the geometric sequence. It occurs in applications to finance and population growth. An arithmetic sequence, we found that we added a number to the initial term to form the sequence. In geometric sequences, we start with a number and then generate a sequence by repeatedly multiplying a nonzero number r.

A **geometric sequence*** may be defined recursively as $a_1 = a$, $\frac{a_n}{a_{n-1}} = r$, or as

$$\boxed{a_1 = a, \qquad a_n = ra_{n-1}}$$

where $a_1 = a$ and $r \neq 0$ are real numbers. The number a_1 is the first term, and the nonzero number r is called the **common ratio.**

so $\frac{a_2}{a_1} = r = \frac{a_4}{a_3}$ ← previous term

Identify the first term and common ratio:

$$2, 6, 18, 54, 162, \ldots$$

$a_1 = 2$

$r = \frac{6}{2} = 3$

or

$r = \frac{18}{6} = 3$

Identify the first term and common ratio:

$$\{s_n\} = 2^{-n}$$

$s_1 = 2^{-1} = \frac{1}{2}$

$s_2 = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

$r = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4}\left(\frac{2}{1}\right) = \frac{1}{2} = r$

Identify the first term and common ratio:

$$\{t_n\} = \{3 \cdot 4^n\}$$

$t_1 = 3 \cdot 4^1 = 12$

$t_2 = 3(16) = 48$

$\frac{48}{12} = r = 4$

*n*th Term of a Geometric Sequence

For a geometric sequence $\{a_n\}$ whose first term is a_1 and whose common ratio is r, the nth term is determined by the formula

$$a_n = a_1 r^{n-1} \qquad r \neq 0$$

Given the sequence $10, 9, \frac{81}{10}, \frac{729}{100}, \ldots$

a) Find the nth term
b) find the 9th term
c) find a recursive formula for the sequence

$r = \frac{9}{10}$

also:

$\dfrac{\frac{81}{10}}{9} = \frac{\cancel{81}^{9}}{10} \cdot \frac{1}{\cancel{9}_{1}}$

$r = 9/10$

a) $a_n = a_1 r^{n-1} = 10\left(\frac{9}{10}\right)^{n-1}$

b) $a_9 = 10\left(\frac{9}{10}\right)^8 \approx 4.3046721$ — exact: as calculator does not go out farther.

c) $a_1 = 10$, $a_n = \frac{9}{10}(a_{n-1})$ (definition)

Find the sum of the first n terms of a geometric sequence, (Partial Sum)

Given a geometric sequence, we can calculate the sum of any given number of the terms.

Sum of the First *n* Terms of a Geometric Sequence

Let $\{a_n\}$ be a geometric sequence with first term a_1 and common ratio r, where $r \neq 0, r \neq 1$. The sum S_n of the first n terms of $\{a_n\}$ is

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} = \sum_{k=1}^{n} a_1 r^{k-1}$$

Sum of n terms

Geometric Know

$$\boxed{S_n = a_1 \cdot \frac{1 - r^n}{1 - r}} \qquad r \neq 0, 1$$

Find the sum S_n, for the first n terms of the geometric series:

$$\left\{\left(\frac{1}{2}\right)^n\right\}$$

$S_1 = \left(\frac{1}{2}\right)^1 = 1/2$

$S_2 = \left(\frac{1}{2}\right)^2 = 1/4$

$\frac{1}{2} = r = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{\cancel{4}} \cdot \cancel{2} = \frac{1}{2}$

$S_n = \frac{1}{2} \cdot \dfrac{1 - \frac{1}{2}^n}{1 - \frac{1}{2}} = \frac{1}{\cancel{2}}\left(\dfrac{1 - \frac{1}{2}^n}{\frac{1}{\cancel{2}}}\right)$

$S_n = 1 - \left(\frac{1}{2}\right)^n$

When we think about a geometric series, we realize that it will continue to go on and on. Hence it is called and **Infinite Geometric Series**.

The infinite sum described will do one of two things. First if the sum S_n approaches a number $L \text{ as } n \to \infty$ we say sum of the infinites geometric series **converges.** If the series does not converge to a value it is called a **divergent series.**

Sum of an Infinite Series

Convergence of an Infinite Geometric Series

If $|r| < 1$, the infinite geometric series $\sum_{k=1}^{\infty} a_1 r^{k-1}$ converges. Its sum is

$$\boxed{\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1-r}}$$

$\Leftarrow$ Sum of whole Sequence

Determine if the geometric series converges or diverges. If it converges, find its sum.

$$\sum_{k=1}^{\infty} 2\left(\frac{2}{3}\right)^{k-1}$$

Converges? yes because

$r = \frac{2}{3} \Rightarrow \left|\frac{2}{3}\right| < 1$

$a_1 = 2$

$$\sum_{K=1}^{\infty} 2\left(\frac{2}{3}\right)^{k-1} = \frac{2}{1-\frac{2}{3}}$$

$$= \frac{2}{\frac{1}{3}} = 2 \cdot 3 = \underline{\underline{6}}$$

Writing a Repeated Decimal as a Fraction:

Show that the repeating decimal $0.999 \ldots = 1$

$a_1 = \frac{9}{10}$

$.999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots$

$(.9 + .09 + .009)$

$r = \frac{\frac{9}{100}}{\frac{9}{10}} = \frac{10}{100} = \frac{1}{10}$

$|.1| < 1$

$$\sum_{K=1}^{\infty} \frac{9}{10}\left(\frac{1}{10}\right)^{K-1} = \frac{\frac{9}{10}}{1-\frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = \underline{\underline{1}}$$

so: Converges

QED :)

Initially, a pendulum swings through an arc of 18 inches. On each successive swing, the length of the arc is 0.98 of the previous length.

a) What is the length of the arc of the 10th swing?
b) On which swing is the length of the arc first less than 12 inches?
c) After 15 swings, what total distance will the pendulum have swung?
d) When it stops, what total distance will the pendulum have swung?

a) 1^{st} swing $= 18$

$2^{nd} = 18(.98)$

$3^{rd} = (18(.98)).98 = 18(.98^2)$ so: $a_1 = 18$, $r = .98$

$a_{10} = a_1(r^9) = 18(.98)^9 = \boxed{15.007 \text{ inches}}$

(b) $18(.98^{n-1}) = 12$

$\log .98^{n-1} = \log 12/18$

$n - 1^{+1} = [\log(12/18) / \log(.98)] + 1$

$n = 21.07 \Rightarrow 21^{st}$ swing pendulum greater than 12 in

so pendulum less 12 in. $\boxed{\text{on } 22^{nd} \text{ swing}}$

(c) Sum of 15 swings: $a_1\left(\frac{1-r^n}{1-r}\right)$

$S_{15} = 18\left(\frac{1-.98^{15}}{1-.98}\right) \approx \boxed{253.3 \text{ inches}}$

(d) total distance? $18(.98) + 18(.98^2) + 18(.98^3) + \ldots$

use sum of infinite series as $|r| < 1$:

$$\sum_{k=1}^{\infty} 18(.98^{k-1}) = \frac{18}{1-.98} = 900 \text{ in}$$