Precalculus

Lesson 12.1: Sequences

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A **sequence** is a function *f* whose domain is the set of positive integers. The values *f*(1), *f*(2), *f*(3), … are called terms.

For Example:

$$-2, \quad 4, \quad 6, \quad 8, \quad \dots \quad , 2n$$

Since we are talking about a function, we can graph sequences.

Write down the first six terms of the following sequence.

$$\{a_n\} = \left\{\frac{n-1}{n}\right\}$$

$a_1 = \frac{1-1}{1} = 0$

$a_2 = \frac{2-1}{2} = \frac{1}{2}$

$a_3 = \frac{3-1}{3} = \frac{2}{3}$

$a_4 = \frac{4-1}{4} = \frac{3}{4}$

$a_5 = \frac{4}{5}$

$a_6 = \frac{5}{6}$

note pattern, predict term 5 & 6 :)

Write down the first six terms of the following sequence.

$$\{b_n\} = \left\{(-1)^{n+1}\left(\frac{2}{n}\right)\right\}$$

$b_1 = -1^2\left(\frac{2}{1}\right) = 2$

$b_2 = -1^3\left(\frac{2}{2}\right) = -1$

$b_3 = -1^4\left(\frac{2}{3}\right) = \frac{2}{3}$

$b_4 = -1^5\left(\frac{2}{4}\right) = -\frac{2}{4} = -\frac{1}{2}$

$b_5 = \frac{2}{5}$

$b_6 = -\frac{2}{6} = -\frac{1}{3}$

Write down the first six terms of the following sequence.

$$\{c_n\} = \begin{pmatrix} n \text{ if } n \text{ is even} \\ \frac{1}{n} \text{ if } n \text{ is odd} \end{pmatrix}$$

$c_1 = \frac{1}{1} = 1$

$c_2 = 2$

$c_3 = \frac{1}{3}$

$c_4 = 4$

$c_5 = \frac{1}{5}$

$c_6 = 6$

Determining a Sequence from a Pattern

Number the terms and see what happens between each term:

1 2 3 4

(a) $\frac{e}{1}, \frac{e^2}{2}, \frac{e^3}{3}, \frac{e^4}{4}, \dots \quad a_n = \frac{e^n}{n}$

1 2 3 4

(b) $\frac{1}{1}, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots \quad b_n = \frac{1}{3^{n-1}}$

$\frac{1}{3^0} \quad \frac{1}{3^1} \quad \frac{1}{3^2} \quad \frac{1}{3^3}$

1 2 3 4

(c) $1, 3, 5, 7, \dots \quad c_n = 2n - 1$

$n = 2 \quad 2(2) = 4$

$n = 3 \quad 2(3) = 6$

$n = 4 \quad 2(4) = 8$

1 2 3 4 5

(d) $1, 4, 9, 16, 25, \dots \quad d_n = n^2$

$2^2 \; 3^2 \; 4^2 \; 5^2$

(e) $\frac{1}{1}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \dots \quad e_n = (-1)^{n+1}\left(\frac{1}{n}\right)$

Factorial

A factorial is a non negative integer written with an exclamation mark. If $n \geq 0$ is an integer, the factorial is defined as follows:

$$0! = 1 \quad and\ 1! = 1$$

$$n! = n(n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \quad for\ n \geq 2$$

and.....

$$n! = n(n-1)!$$

Solve:

$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362{,}880$

$\frac{12!}{10!} = \frac{12 \cdot 11 \cdot \cancel{10!}}{\cancel{10!}} = 132$

$\frac{3!\,7!}{4!} \quad \frac{3 \cdot 2 \cdot 1 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!}} = 1260$

CALCULATOR!

Enter # 9

$\boxed{\text{MATH}}$

◄ PRB

#4 !

$\boxed{\text{ENTER}}$

A Sequence Defined by a Recursive Formula

A second way of defining a sequence is to assign a value to the first term and specify the nth term by a formula or equation that involves on or more of the terms preceding it. The sequence is defined **recursively**, and the formula is a **recursive formula**.

Write the first 5 terms of the recursive sequence

6 typo!

$$u_1 = 1,\ u_2 = 1,\ u_n = u_{n-2} + u_{n-1}$$

$u_1 = 1$

$u_2 = 1$

$u_3 = u_{3-2} + u_{3-1} = u_1 + u_2 = 1+1 = 2$

$u_4 = u_{4-2} + u_{4-1} = u_2 + u_3 = 1+2 = 3$

$u_5 = u_{5-2} + u_{5-1} = u_3 + u_4 = 2+3 = 5$

This is known as the Fibonacci Sequence

terms are Fibonacci numbers

Sigma Notation: a short cut notation to indicate the sum of some or all of the terms of a sequence.

Given a sequence

$$a_1,\ a_2, a_3, a_4, \ldots a_n.$$

we can write the sum of the first n terms using **summation notation**, or **sigma notation.** The notation derives its name from the Greek Letter **Σ.** This corresponds to our S for "sum." The following notation is used

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots a_n$$

k is called the index of summation, it is basically the starting number for the sequence.

Write out each sum

$$\sum_{k=1}^{10} \frac{1}{k} =$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$$

$$\sum_{k=1}^{n} k! = 1! + 2! + 3! + 4! + \cdots + n!$$

Express each sum using summation notation

$$1^2 + 2^2 + 3^2 + \cdots + 9^2$$

1 2 3

$$= \sum_{K=1}^{9} 1^K$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}$$

↑ form

$$= \sum_{K=1}^{n} \frac{1}{2^{K-1}}$$

The following sums are natural consequences of properties of the real numbers. These are useful for adding terms of a sequence algebraically:

Properties of Sequences

If $\{a_n\}$ and $\{b_n\}$ are two sequences and c is a real number, then:

$$\sum_{k=1}^{n} (ca_k) = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c\sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

$$\sum_{k=j+1}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{k=1}^{j} a_{k,} \quad \text{where } 0 < j < n$$

The formulas for sums of powers of the first *n* natural numbers are important in calculus:

Formulas for Sums of Sequences

$$\sum_{k=1}^{n} c = \underbrace{c + c + \cdots + c}_{n \text{ terms}} = cn \qquad c \text{ is a real number}$$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Find the sums:

$$\sum_{k=1}^{5}(3k) = 3\sum_{K=1}^{5} K$$

$$= 3\,\frac{(5)(5+1)}{2}$$

$$= \frac{3(5)\cancel{(6)}^{3}}{\cancel{2}} = (3)(3)(5) = \underline{\underline{45}}$$

$$\sum_{k=1}^{10}(k^3+1) = \sum_{K=1}^{10} K^3 + \sum_{K=1}^{10} 1$$

$$= \left[\frac{10(10+1)}{2}\right]^2 + 1(10)$$

$$= \left[\frac{(\cancel{10}^{5})(11)}{\cancel{2}_{1}}\right]^2 + 10$$

$$55^2 + 10 = 3025+10 = \underline{\underline{3035}}$$

$$\sum_{k=1}^{24}(k^2-7k+2) = \sum_{K=1}^{24} K^2 - 7\sum_{K=1}^{24} K + \sum_{K=1}^{24} 2$$

$$= \frac{(\cancel{24}^{4})(24+1)(48+1)}{\cancel{6}_{1}} - \frac{7(\cancel{24}^{12})(25)}{\cancel{2}_{1}} + 2(24)$$

$$= (4)(25)(49) - 7(12)(25) + 48$$

$$= 4900 - 2100 + 48$$

$$= \underline{\underline{2848}}$$

$$\sum_{k=6}^{20}(4k^2) = 4\left[\sum_{K=1}^{20} K^2 - \sum_{K=1}^{5} K^2\right]$$

$$= 4\left[\frac{20(20+1)(40+1)}{6} - \frac{5(5+1)(10+1)}{6}\right]$$

$$= 4\left[\frac{\overset{10}{\cancel{20}}\,\overset{7}{\cancel{(21)}}(41)}{\underset{\cancel{3}\ 1}{\cancel{6}}} - \frac{5(\cancel{6})(11)}{\cancel{6}}\right]$$

$$= 4\left[10(7)(41) - 55\right]$$

$$= 4\left[2870 - 55\right] = \underline{\underline{11,260}}$$