Precalculus

Lesson 11.5: Partial Fraction Decomposition

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When we add fractions together we need to have a common denominator:

$$\frac{2}{x-1}+\frac{1}{2x+1}=\frac{2(2x+1)+(x-1)}{(x-1)(2x+1)}=\frac{3x+1}{2x^2-x-1}$$

Well, in some applications of algebra and calculus, we will need to reverse this process. Basically, we will need to take a fraction and express it as the sum of simpler fractions. This process is called **partial fraction decomposition.**

Non-repeated Linear Factors: The denominators have distinct linear factors, with no factor repeated.

Steps:
1. Factor the denominator into its linear factors.
2. Then write the expression as fractions with one factor for each of the denominator. Since we don't know what the numerators are, assign variables (capital letters) for the unknown numerators.
3. Multiply each side of the equation through by the common denominator. This gets rid of all the denominators.
4. Simplify the equation by distributing the coefficients, then group like terms together (linear terms together and constants together).
5. For the 2 sides to be equal the coefficients of the 2 polynomials must be equal. "Equate the coefficients."
6. Solve for A, Band C.

$$\frac{x}{x^2-5x+6}=$$

$$\left(\frac{x}{\cancel{(x-3)}\cancel{(x-2)}}\right)(x-3)(x-2)=\left(\frac{A}{\cancel{x-3}}+\frac{B}{\cancel{x-2}}\right)(x-3)(x-2)$$

$$x = A(x-2)+B(x-3)$$

$$1x+0 = Ax-2A+Bx-3B$$

$$2(1=A+B)$$

$$0=-2\cancel{A}-3B$$

$$2=2\cancel{A}+2B$$

$$2=-B$$

$$B=-2$$

$$1=A-2$$

$$A=3$$

$$\therefore \boxed{=\frac{3}{x-3}-\frac{2}{x-2}}$$

Non-repeated Irreducible Quadratic Factors: The denominator contains a quadratic factor that is not factorable. Here the corresponding partial fraction decomposition will have the form:

$$\frac{P(x)}{Q(x)} = \frac{Ax+B}{ax^2+bx+c}$$

1. Factor denominator if necessary
2. Clear fractions by multiplying each side with the left side's denominator.
3. Expand and combine like terms.
4. Equate coefficients to solve for the variables.

$$\frac{3x-5}{x^3-1} = \frac{3x-5}{(x-1)(x^2+x+1)} = \left[\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}\right]$$

$(x-1)(x^2+x+1)$

$(x-1)(x^2+x+1)$

$$3x-5 = A(x^2+x+1) + (Bx+C)(x-1)$$

$$3x-5 = Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$$

$A+B=0$ $\qquad 3 = A - B + \cancel{C}$ $\qquad -5 = A - C$

$A = -B$ $\qquad -5 = A \quad - \cancel{C}$ $\qquad -5 = -\frac{2}{3} - C$

$-2 = 2A - B$

$-2 = -2B - B$ $\qquad C = 5\left(\frac{3}{3}\right) - \frac{2}{3}$

$-2 = -3B$ $\qquad C = \frac{15-2}{3}$

$\frac{2}{3} = B$, $\quad A = \frac{-2}{3}$ $\qquad C = \frac{13}{3}$

$$\boxed{= \frac{-2/3}{x-1} + \frac{2/3\,x + \frac{13}{3}}{x^2+x+1}}$$

Repeated Quadratic Factors: Sometimes we will have a factorization that contains a quadratic factor that cannot be factored. The process is a combination of the previous two examples.

$$\frac{x^3+x^2}{(x^2+4)^2} = \left(\frac{Ax+B}{\cancel{(x^2+4)}} + \frac{Cx+D}{(x^2+4)^2}\right)(x^2+4)^2$$

$$x^3+x^2 = (Ax+B)(x^2+4) + Cx + D$$

$$x^3+x^2 = Ax^3 + 4Ax + Bx^2 + 4B + Cx + D$$

$A=1$ $\quad B=1$ $\quad 4A+C=0$ $\quad 4B+D=0$

$4+C=-4$ $\quad 4+D=0$

$C=-4$ $\quad D=-4$

$$\Rightarrow \boxed{\frac{x+1}{(x^2+4)} + \frac{-4x-4}{(x^2+4)^2}}$$