Precalculus

Lesson 5.8: Exponential Growth and Decay Models: Newton's Law, Logistic Growth and Decay

Mrs. Snow, Instructor

Many natural phenomena have been found to follow the law that an amount A varies with time t according to the function:

$$A(t) = A_0 e^{kt}$$

A_0 = the original amount at $t = 0$
k = growth rate
where: $k > 0$ indicates growth and $k < 0$ indicated decay

For modeling cell growth for N number of cells:

$$N(t) = N_0 e^{kt} \qquad k > 0$$

Bacterial Growth

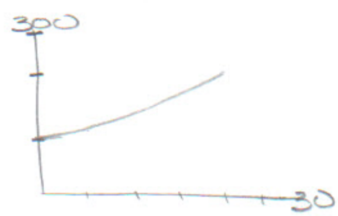
A colony of bacteria grows according to the law of uninhibited growth according to the function $N(t) = 100e^{0.045t}$, where N is measured in grams and t is measured in days.

(a) Determine the initial amount of bacteria.
(b) What is the growth rate of the bacteria?
(c) Graph the function using a graphing utility.
(d) What is the population after 5 days?
(e) How long will it take for the population to reach 140 grams?
(f) What is the doubling time for the population?

(a) at $t = 0 \rightarrow N_0 = 100$ grams

(b) what is K? $K = .045 = 4.5\%$

(c)



(d) $x = 5$ d.y $= 125.2$ g

(e) on graph set $y_2 = 140$

$140 = 100e^{.045t}$

$\ln \frac{140}{100} = \ln e^{.045t}$

$\frac{\ln 1.4}{.045} = \frac{.045t}{.045}$

$t \sim 7.5$ days

(f) $200 = 100e^{.045t}$

$\ln 2 = \ln e^{.045t}$

$\frac{\ln 2}{.045} = \frac{.045t}{.045}$

15.4 days $\approx t$

Uninhibited Radioactive Decay

The growth/decay formula may be applied to radioactive decay:

The amount A of a radioactive material present at time t is given by:

$$A(t) = A_0 e^{kt} \qquad k < 0$$

A_0 =the original amount of radioactive material
k =rate of decay, a negative number
t =time

Estimating the Age of Ancient Tools

Traces of burned wood along with ancient stone tools in an archeological dig in Chile were found to contain approximately 1.67% of the original amount of carbon 14.

(a) If the half-life of carbon 14 is 5700 years, approximately when was the tree cut and burned? $\uparrow t$ for $\frac{1}{2}A_0$

(b) Using a graphing utility, graph the relation between the percentage of carbon 14 remaining and time.

(c) Use a graphing utility to determine the time that elapses until half of the carbon 14 remains. This answer should equal the half-life of carbon 14.

(d) Use a graphing utility to verify the answer found in part (a).

(a) $.0167 A_0 = A_0 e^{kt}$ first find "k" $\frac{1}{2}\cancel{A_0} = \cancel{A_0} e^{k(5700)}$

$\ln\frac{1}{2} = \ln e^{5700k}$

$\ln\frac{1}{2} = 5700k \cancel{\ln e}$

$-1.216 \times 10^{-4} = k$

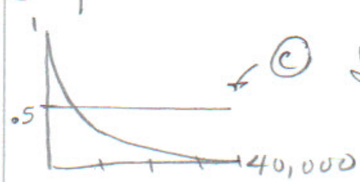
or $-.0001216 = k$

$.0167 \cancel{A_0} = \cancel{A_0} e^{-.0001216t}$

$\ln .0167 = -.0001216t \cancel{\ln e}$

$\sim 33{,}654$ years

(b) $y = e^{-.0001216t}$ (c) $y_2 = .5$



(c) Intersection at $t = 5700$ yr.

(d) 2^{nd} trace (calculate); 1 (value); $x = 33654$

$y = .0167$

Newton's Law of Cooling

Newton's Law of Cooling stated that the temperature of a heated object decreases exponentially over time toward the temperature of the surrounding medium:

The temperature u of a heated object at a given time t can be modeled by:

$$u(t) = T + (u_0 - T)e^{kt} \qquad k < 0$$

$u(t)$ =temperture

T is the constant temperature of the surrounding medium

$u_0$is the initial temperature of the heated object

k = negative constant

t =time

An object is heated to 100°and is then allopwed to cool in a room whose air temperature is 30°C. T $\leftarrow u_0$ $u(t)$ t

- If the temperature of the object is 80°Cafter 5 minutes, when will its temperature be 50°C? $u(t)$
- Graph the relation found between the temperature and time
- Determine the elapsed time before the object is 35°C

#1 find K

$$80 = 30 + (100-30)e^{K5}$$

$$50 = 70e^{K5}$$

$$\ln\frac{50}{70} = \ln e^{5k}$$

$$\ln 5/7 = 5k \cancel{\ln e}$$

$$\frac{\ln 5/7}{5} = K$$

$$\sim -.06729$$

$$\sim -.0673$$

time for temp at 50°

$$50 = 30 + 70e^{-.0673t}$$

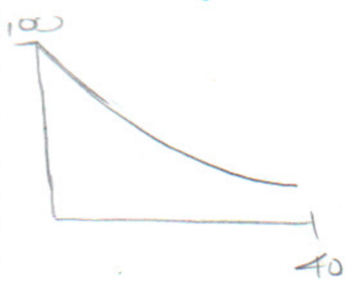
$$\ln\frac{20}{70} = \ln e^{-.0673t}$$

$$\ln 2/7 = -.0673t \cancel{\ln e}$$

$$18.6\text{ min} = t$$

Graph:

$$u(t) = 30 + 70e^{-.0673t}$$



35°C

$$35 = 30 + 70e^{-.0673t}$$

$$\ln\frac{5}{70} = \ln e^{-.0637t}$$

$$\frac{\ln 5/70}{-.0673} = t$$

$$\sim 39.21\text{ min}$$

Logistic Model

If we look back at the growth model for cell, we realize that the formula allows for unlimited growth. We know that cell division eventually is limited by factors such as living space and food supply. This idea is called **carrying capacity.** The **logistic model** may be used to describe situations where the growth or decay of the dependent variable is limited.

The population P after time t is given by:

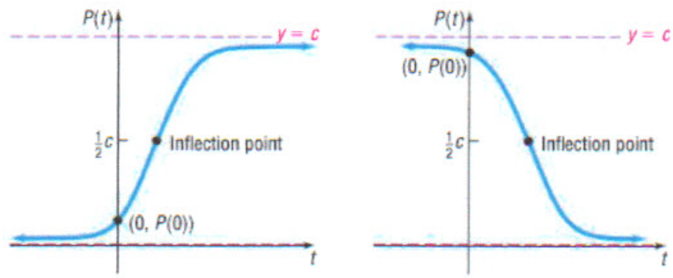
$$P(t) = \frac{c}{1 + ae^{-bt}}$$

$a, b, and\ c$ = constants with $a > 0\ and\ c > 0$

$b > 0$ growth

$b < 0$ decay

The general graph looks like:



Properties of the Logistic Model, Equation (5)

1. The domain is the set of all real numbers. The range is the interval $(0, c)$, where c is the carrying capacity.
2. There are no x-intercepts; the y-intercept is $P(0)$.
3. There are two horizontal asymptotes: $y = 0$ and $y = c$.
4. $P(t)$ is an increasing function if $b > 0$ and a decreasing function if $b < 0$.
5. There is an **inflection point** where $P(t)$ equals $\frac{1}{2}$ of the carrying capacity.

   The inflection point is the point on the graph where the graph changes from being curved upward to curved downward for growth functions and the point where the graph changes from being curved downward to curved upward for decay functions.
6. The graph is smooth and continuous, with no corners or gaps.

Fruit Fly Population

Fruit flies are placed in a half-pint milk bottle with a banana (for food) and yeast plants (for food and to provide a stimulus to lay eggs). Suppose that the fruit fly population after t days is given by: $P(t) = \frac{230 \leftarrow c}{1+56.5e^{-0.37t} \leftarrow b}$

a. State the carrying capacity and the growth rate.
b. Determine the initial population.
c. What is the population after 5 days?
d. How long does it take for the population to reach 180?
e. Determine how long it takes for the population to reach one-half of the carrying capacity.

(a) carrying capacity $= C = 230$ fruit flies
growth rate $= b = |-.37| = .37 = 37\%$

(b) $t=0$ population $P(0) = \frac{230}{1+56.5\,e^{-.37(0)}}$

$= \frac{230}{1+56.5} = 4$ flies

(c) $t=5$ $P(5) = \frac{230}{1+56.5\,e^{-.37(5)}} = 23$ flies

(d) population $=180$ $\quad 180 = \frac{230}{1+56.5e^{-.37t}}$

$\cancel{1}+56.5e^{-.37t} = \frac{230}{180} - 1$

$\ln e^{-.37t} = \ln \frac{\frac{230}{180} - 1}{56.5}$

$\frac{-.37t}{-.37} = \frac{\ln \boxed{\frac{\left(\frac{230}{180}-1\right)}{56.5}}}{\cancel{.37}}$ $.37$

$t = 14.4$ days

(e) $P(t)=115$

$115 = \frac{230}{1+56.5e^{-.37t}}$

$\cancel{1}+\frac{56.5e^{-.37t}}{56.5} = \frac{\frac{230}{115}-1}{56.5}$

$-.37t \quad \ln e^{-.37t} = \frac{\ln \boxed{\frac{\frac{230}{115}-1}{56.5}}}{-.37}$

$\boxed{t = 10.9 \text{ days}}$