Precalculus
Lesson 5.1: Composite Functions
Mrs. Snow, Instructor

Composite Functions: A composite function is a function that is made or composed of more than one "independent" function. In general, a number *x* is applied to one function the result or output is then applied to a second function.

Given two functions f and g, the **composite function,** denoted by $f \circ g$ (read as "f composed with g"), is defined by

$$\boxed{(f \circ g)(x) = f(g(x))}$$

The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.

Look at domain of $g(x)$ → restrictions?

Domain of a composite: combine with the domain of $f(g(x))$ → restrictions?

The domain of a composite function, $f\ o\ g$, if defined whenever both $g(x)$ and $f(g(x))$ are defined.

Evaluating a composite function

Suppose that $f(x) = 2x^2 - 3$ and $g(x) = 4x$. Find:

(a) $(f \circ g)(1)$ (b) $(g \circ f)(1)$ (c) $(f \circ f)(-2)$ (d) $(g \circ g)(-1)$

(a)
$f(g(1)) =$
$g(1) = 4(1) = 4$
$f(4) = 2(4^2) - 3$
$= 32 - 3$
$= 29$

(b)
$g(f(1)) =$
$f(1) =$
$2(1^2) - 3 = -1$
$g(-1) = 4(-1)$
$= -4$

(c)
$f(f(-2)) =$
$f(-2) =$
$2(-2)^2 - 3 =$
$8 - 3 = 5$
$f(5) =$
$2(5)^2 - 3 =$
$50 - 3 = 47$

(d)
$g(g(-1)) =$
$g(-1) = 4(-1)$
$= -4$
$g(-4) =$
$4(-4) =$
$= -16$

Finding a composite function and its domain

Suppose that $f(x) = x^2 + 3x - 1$ and $g(x) = 2x + 3$.

Find: (a) $f \circ g$ (b) $g \circ f$

Then find the domain of each composite function.

Domain

$f(x) = \mathbb{R}$

$g(x) = \mathbb{R}$

$f(g(x)) = (2x+3)^2 + 3(2x+3) - 1$

$= 4x^2 + 12x + 9 + 6x + 9 - 1$

$= 4x^2 + 18x + 17$ f(g(x)) Domain $\mathbb{R}$

combine

← page 1

∴ f(g(x)) & g(x) All Real

$g(f(x)) = 2(x^2 + 3x - 1) + 3$

$= 2x^2 + 6x - 2 + 3$

$= 2x^2 + 6x + 1$ Domain $\mathbb{R}$ ∴

combine

Domain

f(g(x)) & f(x)

$= \mathbb{R}$

Suppose that $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{4}{x-1}$.

Find: (a) $f \circ g$ (b) $f \circ f$

Then find the domain of each composite function.

Domain

$f(x)$ $\{x \mid x \neq -2\}$

$g(x)$ $\{x \mid x \neq 1\}$

Domain f∘g

Both f(g(x)) & g(x)

$$f(g(x)) = \frac{1}{\left(\frac{4}{x-1}\right) + 2\frac{(x-1)}{(x-1)}} = \frac{1}{\frac{4 + 2x - 2}{x-1}}$$

$$= \frac{1}{\frac{2x+2}{x-1}} = \frac{x-1}{2x+2}$$

$2x + 2 \neq 0$

$2x \neq -2$

$x \neq -1$

$\{x \mid x \neq -1, 1\}$

$$f(f(x)) = \frac{1}{\left(\frac{1}{x+2}\right) + 2\left(\frac{x+2}{x+2}\right)} = \frac{1}{\frac{1 + 2x + 4}{x+2}}$$

∴ Domain f∘f

$$= \frac{1}{2x+5}$$ $\Rightarrow 2x + 5 \neq 0$

$2x \neq -5$

$x \neq \frac{-5}{2}$

$\{x \mid x \neq -2, -\frac{5}{2}\}$

Show that two composite functions are equal

If $f(x) = 3x - 4$ and $g(x) = \frac{1}{3}(x + 4)$, show that

$$(f \circ g)(x) = (g \circ f)(x) = x$$

for every x in the domain of $f \circ g$ and $g \circ f$.

$$f(g(x)) = 3\left(\tfrac{1}{3}(x+4)\right) - 4$$
$$= \cancel{3}\left(\tfrac{1}{\cancel{3}}x + \tfrac{4}{3}\right) - 4$$
$$= x + \cancel{4} - \cancel{4}$$
$$= x$$

$$g(f(x)) = \tfrac{1}{3}\left((3x - \cancel{4}) + \cancel{4}\right)$$
$$= \tfrac{1}{3}(3x)$$
$$= x$$

$$\therefore f \circ g = g \circ f = x$$

Finding the components of a composite function

Find functions f and g such that $f \circ g = H$ if $H(x) = (x^2 + 1)^{50}$.

$H(x) = \quad ^{50}$ $\qquad f(g(x))$

$f(x) = x^5$

$g(x) = \quad = x^2 + 1$

Find functions f and g such that $f \circ g = H$ if $H(x) = \dfrac{1}{x+1}$.

$$H(x) = \frac{1}{\quad}$$

$$f(x) = \frac{1}{x}$$

$$g(x) = \quad = x+1$$

Precalculus

Lesson 5.2: One to One Functions; Inverse Functions

Mrs. Snow, Instructor

A quick definition review: A function is a special relation for which every element of the domain corresponds to exactly one element of the range. Now, for a function to be considered **one-to-one or it may be written as "1-1,"** it must also meet the following criteria: every element of the range corresponds to exactly one element of the domain. Another way to look at this is that each ***x*** in the domain has one and only one corresponding point in the range.

A function is one-to-one if any two different inputs in the domain correspond to two different outputs in the range. That is

$$f(x_1) \neq f(x_2) \quad for\ x_1 \neq x_2$$

So, x is unique
no repeaters
and y is unique
no repeaters

Graphically we can determine a one-to-one relationship by using the **horizontal-line-test** to determine of *f* is one-to-one. Basically, this is analogous to the vertical line test, only horizontal.

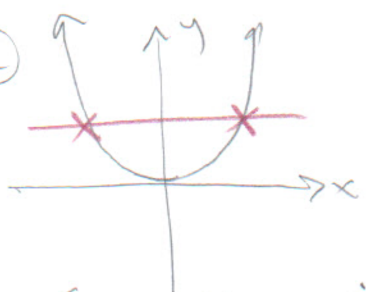
Horizontal-line Test

If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

Using the graph of the function to determine if the functions are 1-1

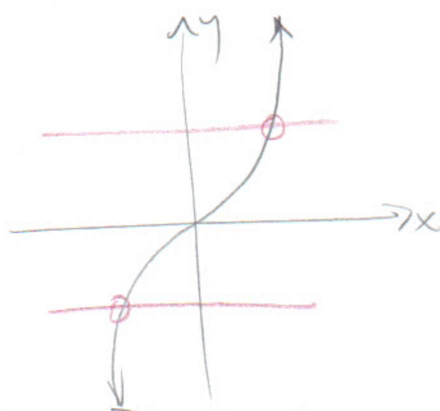
(a) $f(x) = x^2$ (b) $g(x) = x^3$

a



fails Horizontal

not 1-1.



*notice

functions increasing over an interval I are 1-1

& functions decreasing over I are 1-1

Passes Horizontal line test

1-1

Inverses: Another way of saying inverse is opposite. Did you ever play "opposite day" with your parents? If you remember you probably would not confess it; Yes means No and No means Yes! Mathematically: $x \text{ is } y \text{ and } y \text{ is } x$.

$f(x)$ and $g(x)$ are inverses if and only if: when inverses composites $= x$

* $f(g(x)) = x \quad \text{and} \quad g(f(x)) = x$

Suppose that f is a one-to-one function. Then, to each x in the domain of f, there is exactly one y in the range (because f is a function); and to each y in the range of f, there is exactly one x in the domain (because f is one-to-one). The correspondence from the range of f back to the domain of f is called the **inverse function of *f*.** The symbol f^{-1} is used to denote the inverse of f.

Find the inverse of the following one-to-one function:

$f(x)$ $\{(-3, -27), (-2, -8), (-1, -1), (0, 0), (1, 1), (2, 8), (3, 27)\}$

"X" is "y" & "y" is "x"

$f^{-1}(x)$ $\{(-27, -3), (-8, -2)\ (-1, -1), (0, 0)\ (1, 1), (8, 2), (27, 3)\}$

Domain of $f(x)$ is Range of $f^{-1}(x)$

Range of $f(x)$ is Domain $f^{-1}(x)$

(a) Verify that the inverse of $g(x) = x^3$ is $g^{-1}(x) = \sqrt[3]{x}$.

(b) Verify that the inverse of $f(x) = 2x + 3$ is $f^{-1}(x) = \frac{1}{2}(x - 3)$.

(a)

$g(x) = x^3$

$y = x^3$ switch x & y

$x = y^3$ solve for y

$\sqrt[3]{x} = \sqrt[3]{y^3}$

$y = \sqrt[3]{x} = g^{-1}(x)$ ✓

also:

$g(g^{-1}(x)) = (\sqrt[3]{x})^3$
$= x$

$g^{-1}(g(x)) = \sqrt[3]{x^3}$
$= x$

(b)

$f(x) = 2x + 3$

$y = 2x + 3$
switch

$x = 2y + 3$
solve for y

$x - 3 = 2y$

$y = \frac{x-3}{2}$

$y = \frac{1}{2}(x - 3)$
$= f^{-1}(x)$ ✓

D: $x \neq 1$ D: $x \neq 0$

Verify that the inverse of $f(x) = \dfrac{1}{x-1}$ is $f^{-1}(x) = \dfrac{1}{x} + 1$. For what values of x is $f^{-1}(f(x)) = x$? For what values of x is $f(f^{-1}(x)) = x$? *

$y = \dfrac{1}{x-1}$

switch

$x = \dfrac{1}{y-1}$

solve for y

$(y-1)(x) = 1$

$(y-1) = \dfrac{1}{x}$

$y = \dfrac{1}{x} + 1$

$= f^{-1}(x)$ ✓

$f^{-1}(f(x)) = \dfrac{1}{\left(\dfrac{1}{x-1}\right)} + 1$

$= \dfrac{x-1}{1} + 1$

$= x$

$x \neq 1$

domain Restriction

$f(f^{-1}(x)) =$

$\dfrac{1}{\left(\frac{1}{x} + 1\frac{x}{x}\right) - 1} =$

$\dfrac{1}{\left(\frac{1+x}{x}\right) - 1\frac{(x)}{(x)}} =$

$\dfrac{1}{\frac{1+\cancel{x-x}}{x}} =$

$\dfrac{1}{\frac{1}{x}} = x$

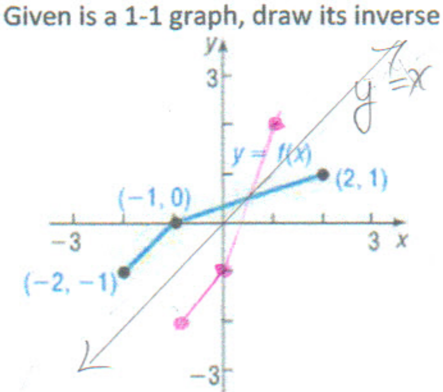
D $\{x \mid x \neq 0\}$

Given is a 1-1 graph, draw its inverse

x	y
-2	-1
-1	0
2	1

switch

x	y
-1	-2
0	-1
1	2



graphically:

Inverses are symmetric about line $y = x$

Find the inverse of $f(x) = 2x + 3$. Graph f and f^{-1} on the same coordinate axes.

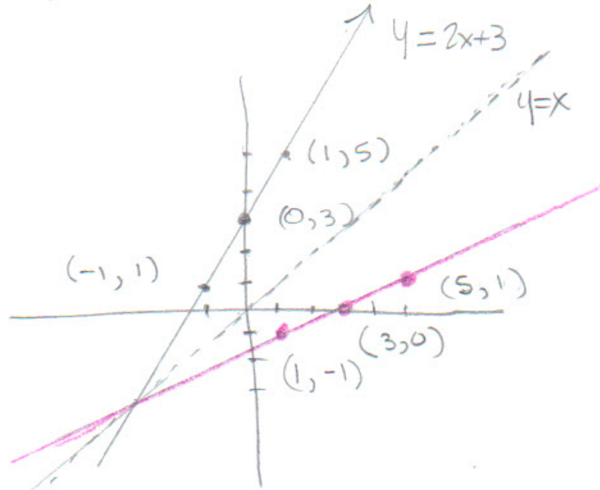
$y = 2x+3$

$x = 2y+3$

$x-3 = 2y$

$\frac{x-3}{2} = y = f^{-1}(x)$

$\frac{1}{2}(x-3) = y$



The following function is one-to-one. Find its inverse and check the result.

$$f(x) = \frac{2x+1}{x-1}, x \neq 1$$

$y = \frac{2x+1}{x-1}$

$x = \frac{2y+1}{y-1}$

$(y-1)(x) = (2y+1)$

$xy - x = 2y+1$

$xy - 2y = (x+1)$

$y(x-2) = (x+1)$

$\Rightarrow y = \frac{(x+1)}{(x-2)} = f^{-1}(x)$

$f(f^{-1}(x)) \overset{?}{=} x$

$$\frac{2\left(\frac{x+1}{x-2}\right)+1}{\left(\frac{x+1}{x-2}\right)-1} = \frac{\frac{2x+\cancel{2}+x-\cancel{2}}{x-2}}{\frac{\cancel{x}+1-(\cancel{x}-2)}{x-2}} = \frac{\frac{3x}{\cancel{(x-2)}}}{\frac{3}{\cancel{(x-2)}}}$$

$= \frac{3x}{3} = x$ ✓

$\& \; f^{-1}(f(x)) = \frac{\left(\frac{2x+1}{x-1}\right)+1}{\left(\frac{2x+1}{x-1}\right)-2} = \frac{\frac{2x+\cancel{1}+x-\cancel{1}}{\cancel{(x-1)}}}{\frac{\cancel{2x}+1-\cancel{2x}+2}{\cancel{(x-1)}}}$

$= \frac{3x}{3} = x$ ✓

By restricting the domain of a function that is not 1-1, we can make the function 1-1 and find its inverse.

Find the inverse of $y = f(x) = x^2$ if $x \geq 0$. Graph f and f^{-1}.

$y = x^2 \quad x \geq 0$

Switch

$x = y^2 \quad y \geq 0$

$\sqrt{x} = \sqrt{y^2}$

$\sqrt{x} = y \quad y \geq 0$