Precalculus
Lesson 4.1 Polynomial Functions and Models
Mrs. Snow, Instructor

Let's review the definition of a polynomial.

A polynomial function of degree n is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and

- The numbers $a_0, a_1, a_2, \dots a_n$ are called coefficients of the polynomial.
- The number a_0 is the constant term.
- The number a_n , the coefficient of the highest power, is the leading coefficient.
- The **degree** of the polynomial function is the largest power of *x* that appears.

Identify the polynomial functions, state degree.

$f(x) = 3x - 4x^3 + x^8$

Polynomial- 8th degree

$g(x) = \dfrac{x^2+3}{x-1}$

NO

$h(x) = 5x^0$

Polyn- Zero degree

$F(x) = (x-3)(x+2)$

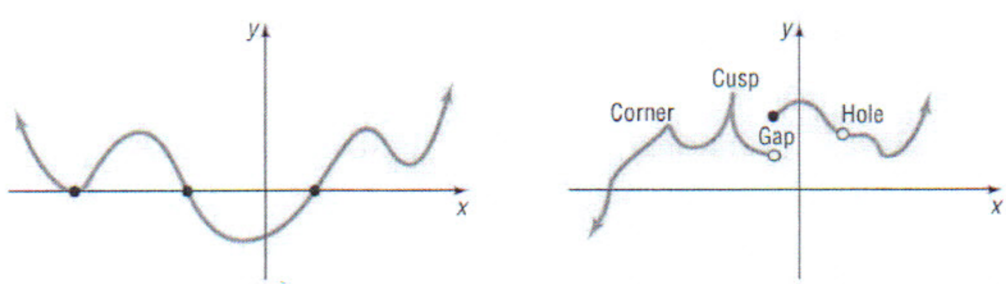
Polyn- 2nd degree

$G(x) = 3x - 4x^{-1}$

NO

$H(x) = \frac{1}{2}x^3 - \frac{2}{3}x^2 + \frac{1}{4}x$

polyn - 3rd degree



(a) Graph of a polynomial function: smooth, continuous

(b) Cannot be the graph of a polynomial function

The textbook defines a **power function** as a monomial function (a single termed polynomial).

$$f(x) = ax^n$$
$$a \text{ is a real nonzero number, and } n > 0$$

There are several basic polynomial functions we need to know.

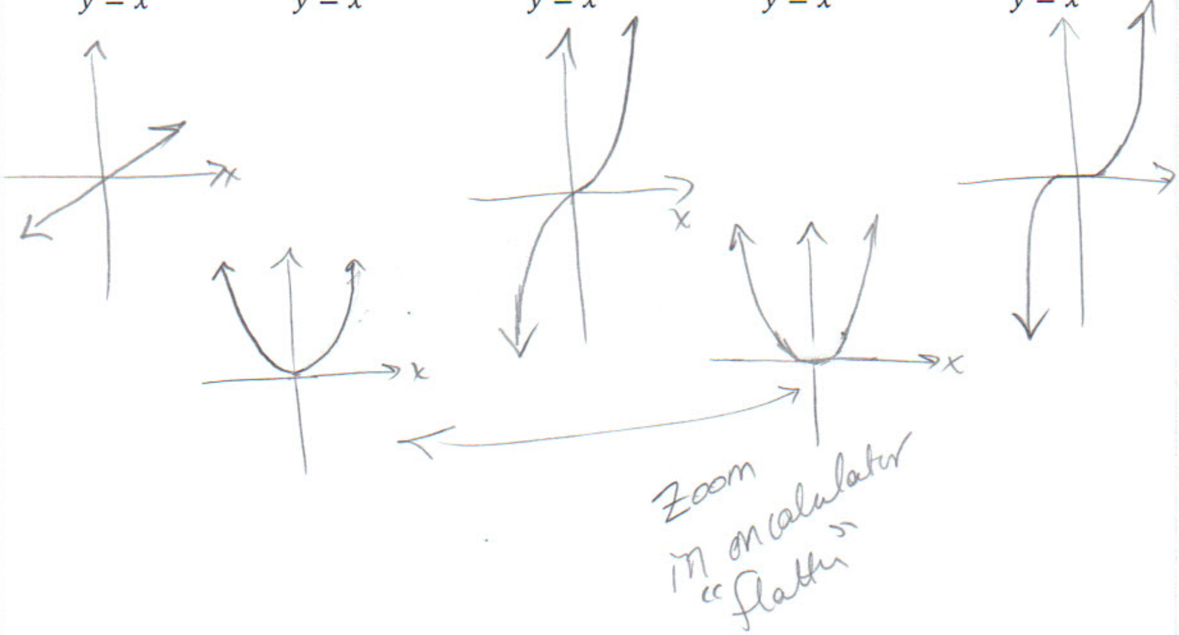
$f(x) = 3x$	$f(x) = -5x^2$	$f(x) = 8x^3$	$f(x) = -5x^4$
degree 1	degree 2	degree 3	degree 4

What is the significance of the leading coefficient?

Vertical stretch, negative = reflect over x-axis

Common power functions Parent functions

$y = x$ $\quad y = x^2$ $\quad y = x^3$ $\quad y = x^4$ $\quad y = x^5$



Transformation of Monomials

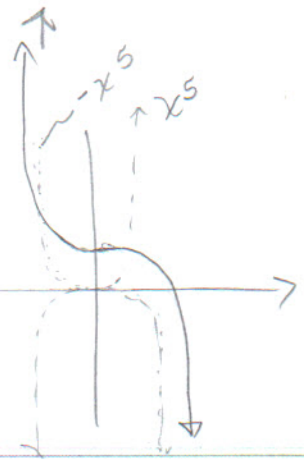
- Sketch the graphs of the following functions,
- state whether the function is even, odd or neither.
- Determine the domain and range
- Determine the intervals over which the function is increasing and decreasing. ID local maxima or minima

$$f(x) = 1 - x^5$$

$= -x^5 + 1$

(−) reflect over x-axis

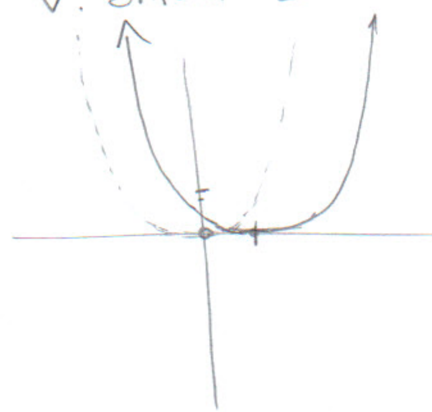
+1



$$f(x) = \frac{1}{2}(x-1)^4$$

$\rightarrow 1$

V. shrink $\frac{1}{2}$



Zeros and Multiplicities

When we look for the **zeros** of a polynomial equation, we are looking for those values of **x** that are solutions to the equation or $P(x) = 0$. Graphically, we see the zeros where the graph crosses or touches the x-axis.

Real Zeros of Polynomials

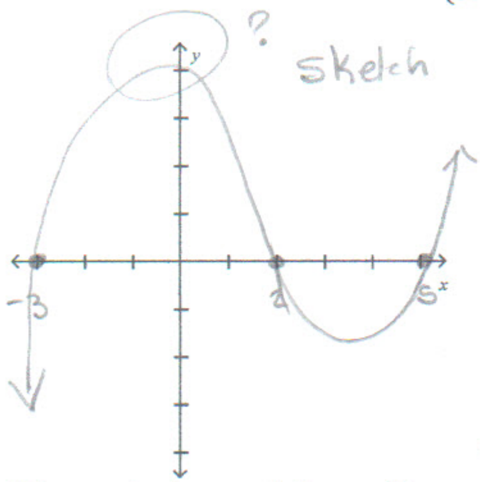
- If f is a polynomial and r is a real number, then the following are equivalent:
  - r is a zero of f.
  - $x = r$ is a solution of the equation $f(x) = 0$.
  - $x - r$ is a factor of $f(x)$.
  - $x = r$ is an x-intercept of the graph of f.

Using Zeros to Graph a Polynomial Function

Sketch Zeros to graph a Polynomial Function of degree 3.

$(zeros = -3, 2, and\ 5)$

degree 3



What are the factors of this graph?

$x = -3$ $\quad x = 2$ $\quad x = 5$

$x + 3 = 0$ $\quad x - 2 = 0$ $\quad x - 5 = 0$

What is the polynomial function? (Yes! multiply out the factors)

$$f(x) = (x+3)(x-2)(x-5)$$
$$= (x+3)(x^2 - 7x + 10)$$
$$x^3 - 7x^2 + 10x$$
$$+3x^2 - 21x + 30$$
$$f(x) = x^3 - 4x^2 - 11x + 30$$

← y-intercept $= 30$

Remember you can also take a polynomial function, factor it and then graph. To make this process easier, always remember to look for common factors of each term to factor out.

Shape of a graph near a zero and the Multiplicity of the zero

If $x - r$ occurs more than once, r is called a repeated, or multiple, zero of f.

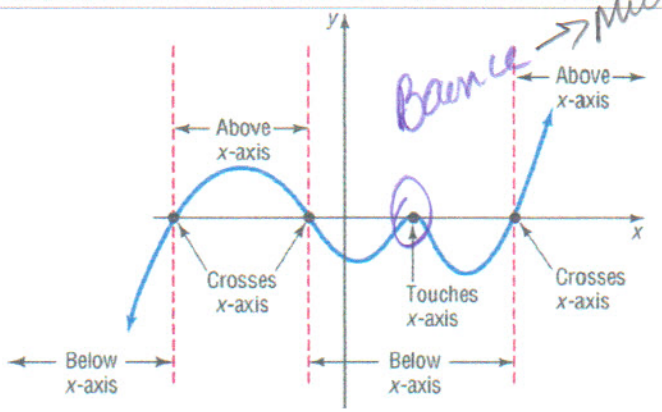
$(x - r)^n$ *

If the zero is a real number, then it will be an x-intercept.

Multiplicity of a zero is EVEN → graph will BOUNCE ON the x-axis at that x-intercept
Multiplicity of a zero is ONE → graph will CROSS the x-axis at that x-intercept
Multiplicity of a zero is ODD other than 1 → graph will WIGGLE ACROSS the x-axis at that x-intercept
The higher the multiplicity the <u>flatter</u> the graph at the zero

* exponent tells us how many times a factor / zero repeated ⇒ n = multiplicity

The graph below has four zeros. Three of which have multiplicities of one and the fourth appears to have a multiplicity of two as we see a bounce.



Identify the zeros and their multiplicities.

$$P(x) = x^4(x-2)^3(x+1)^2 = 0$$

$x^4 = 0$ $\quad (x-2)^3 = 0$ $\quad (x+1)^2 = 0$

$x = 0$ $\quad x - 2 = 0$ $\quad x + 1 = 0$

multip $= 4$ $\quad x = 2$ $\quad x = -1$

multip $= 3$ $\quad$ multip $= 2$

From our last chapter remember our local maximums and minimums; they also known as **extrema** of a polynomial. These are the "hills" or "valleys" where the graph changes from increasing to decreasing or vice versa. ***An extrema is a y-value, not a point.*** !!

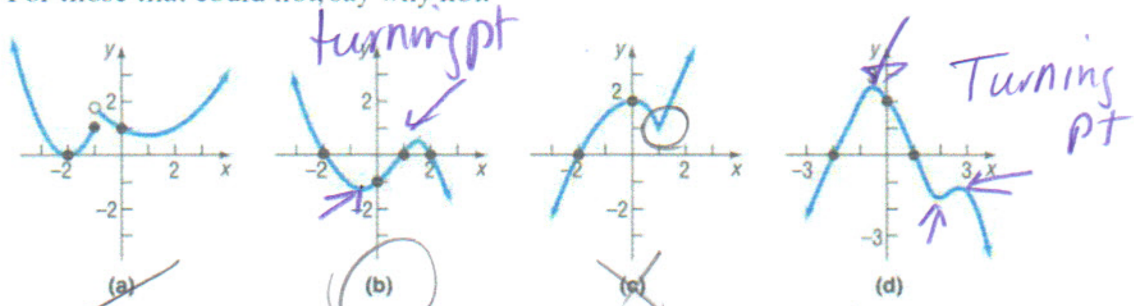
Now at the extrema point, the graph changes from increasing to decreasing or vice versa. This is called a **turning point.**

If $f(x)$ has a degree of n, then the graph of f has at most $\boldsymbol{n-1}$ local extrema.

therefore: degree of n, at most n zeros and $\mathbf{n-1}$ *turning points*

"at most," be careful with this term. A polynomial of degree 5 will have at most 4 extrema or at most 4 turning points. IT MAY NOT HAVE 4 TURNING POINTS OR EXTREMA! Why??? (repeated zeros ⇒ multiplicities)

Which of the graphs in Figure 16 could be the graph of a polynomial function? For those that could, list the zeros and state the least degree the polynomial can have. For those that could not, say why not.



discontinuous

t.p = 2 = n-1
3 = n
degree 3

cusp

t.p = 3 = n-1
4 = n
degree 4

End Behavior

When we graph these polynomials, we put arrows on the end of the curve to show that the graph continues on to infinity. What is happening to the end of the graph? Is the graph rising (increasing) or falling (decreasing)? The **end behavior** of a polynomial is the description of what happens as x approaches infinity (the positive direction) and approaches negative infinity (the negative direction). We have a certain notation use to describe the end behavior.

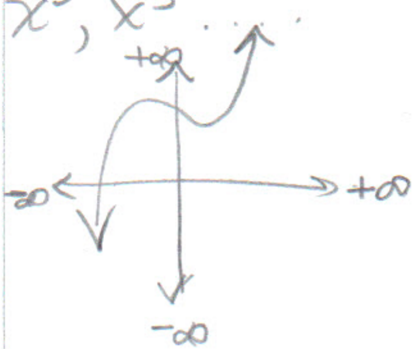
$x \to \infty$ means as x becomes large in the positive direction	$x \to -\infty$ means as x becomes large in the negative direction

For polynomials with degree ≥ 1 *and odd*

If leading coefficient is positive; $a > 0$: implies the graph of f falls to the left and rises to the right.

$$as\ x \to +\infty \quad f(x) \to +\infty$$
$$as\ x \to -\infty \quad f(x) \to -\infty$$

$x^3, x^5 \ldots$

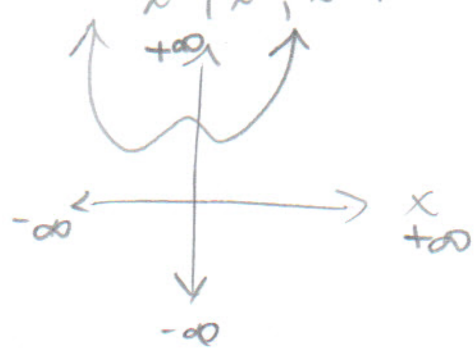


For polynomials with degree ≥ 2 *and even*

If leading coefficient is positive; $a > 0$: implies the graph rises both to the left and right.

$$as\ x \to \pm\infty \quad f(x) \to +\infty$$

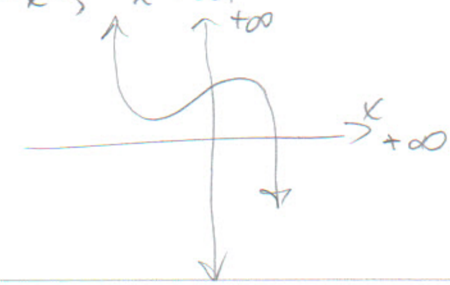
$x^2, x^4, x^6, \ldots$



For polynomials with degree ≥ 1 *and odd*
If leading coefficient is <u>negative</u>; $a < 0$:
implies the graph of f rises to the left and falls to the right.

$$as\ x \to +\infty \quad f(x) \to -\infty$$
$$as\ x \to -\infty \quad f(x) \to +\infty$$

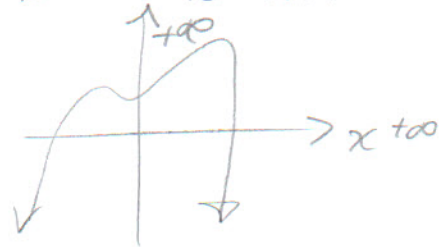
$-x^3, -x^5, \ldots$



For polynomials with degree ≥ 2 *and even*
If leading coefficient is <u>negative</u>; $a < 0$:
implies the graph falls both to the left and right.

$$as\ x \to \pm\infty \quad f(x) \to -\infty$$

$-x^2 \quad -x^4 \ldots\ldots$



Observation:

- *All even-degree polynomials behave, on their ends, like quadratics, and all odd-degree polynomials behave, on their ends, like cubics. In calculus limits are used to convey this idea.*
- *For large values of $|x|$, the graph of the polynomial functions may resemble a power function if the form of ax^n*
- *When a value of a limit approaches infinity, this means the values of the function are **unbounded** in the positive and/or negative direction.*

Analyze a Graph of a Polynomial Function

1. Determine the end behavior of the graph of the function
2. Find the x- and y-intercepts of the graph of the function
3. Determine the zeros of the function and their multiplicity. Use this information to determine whether the graph crosses or bounces the x-axis at each x-intercept.
4. Use a graphing utility to graph the function.
5. Approximate the turning points of the graph
6. Use the information in Steps 1-5 to draw a complete graph of the function by hand.

Analyze the graph of the polynomial function:
$f(x) = (2x+1)(x-3)^2$ ① degree $= 3$ LC is + so

② $(2x+1)(x-3)^2 = 0$ y-inpt is constant of $f(x)$

so $(2x+1)(x-3)(x-3)$

$2x+1 = 0 \quad x-3 = 0$

③ $x = -\frac{1}{2} \quad x = 3$ x-inpt

y intcpt $= 1(-3)(-3) = 9$

multip ⇒ 1 multip ⇒ 2

cross bounce

⑤ $tp = n - 1$
$= 3 - 1$
$= 2$ turning pt

Analyze the graph of the polynomial function:

$f(x) = x^2(x-4)(x+1)$ degree = 4 , lc + so

① $x \to \pm\infty$ $y \to \pm\infty$

② $x^2(x-4)(x+1) = 0$

$x = 0$ $x - 4 = 0$ $x + 1 = 0$

$x = 4$ $x = -1$

multip → 2 multip → 1 multip → 1

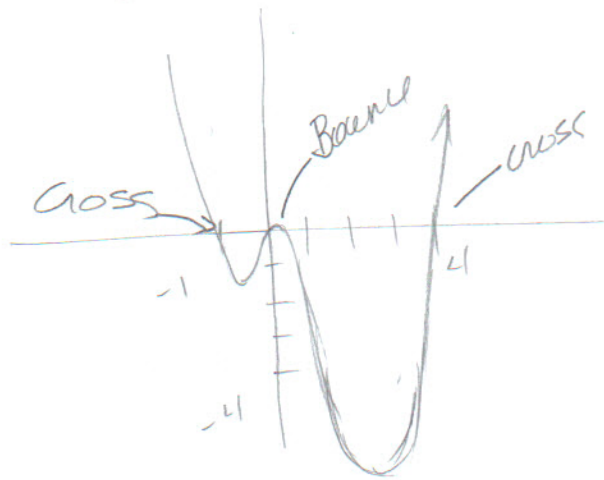
y incpt? $x^2(x-4)(x+1)$

$= x^2(x^2-3x-4)$

$= x^4 - 3x^3 - 4x^2 + 0$

tp = 4 - 1

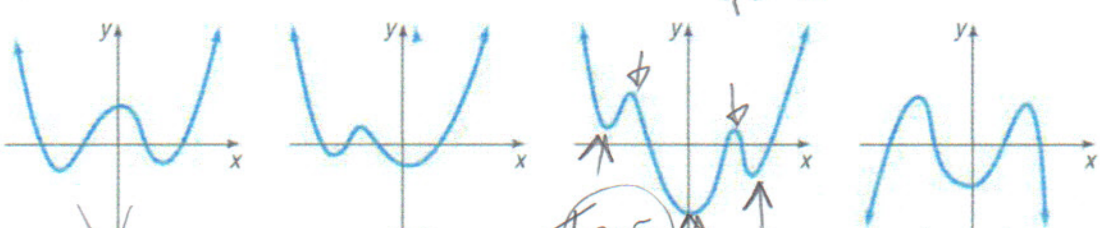
= 3 turning pt



Which graph could model the polynomial and why are the other graphs eliminated?

$f(x) = x^4 + ax^3 + bx^2 - 5x - 6$ + L.C. 4th dg

$y = -6$



(a) (b) (c) (d)

y incpt = positive

4th degree

tp = 4-1

= 3

opens down