# Precalculus
## Lesson 4.2: The Real Zeros of a Polynomial Function
### Mrs. Snow, Instructor

Dividing polynomials is a very similar process to the old long divisions we did in elementary school:

$$\frac{38}{7} = 5 + \frac{3}{7} \; where \, \frac{3}{7} \; is \; the \; fractional \; remainder$$

Proper format for division of polynomials:

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

*or*

$$f(x) = q(x)g(x) + r(x)$$

Dividend

Remainder

Quotient

Divisor

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**Divide:**

$$6x^2 - 26x + 12 \; by \; x - 4$$

$$\begin{array}{r} 6x - 2 \\ x-4 \overline{) \; 6x^2 - 26x + 12} \\ -(6x^2 - 24x) \phantom{+12} \\ \hline -2x + 12 \\ -(-2x + 8) \\ \hline 4 \end{array}$$

4 Remainder

$f(x) =$

$(x-4)(6x-2) + 4$

**Divide:**

0x³ placeholder

$$P(x) = 8x^4 + 6x^2 - 3x + 1$$
$$D(x) = 2x^2 - x + 2$$

$$\begin{array}{r} 4x^2 + 2x \phantom{+6x^2-3x+1} \\ 2x^2-x+2 \overline{) \; 8x^4 + 0x^3 + 6x^2 - 3x + 1} \\ -(8x^4 - 4x^3 + 8x^2) \phantom{-3x+1} \\ \hline 4x^3 - 2x^2 - 3x \phantom{+1} \\ -(4x^3 - 2x^2 + 4x) \phantom{+1} \\ \hline 7x + 1 \end{array}$$

7x + 1 Remainder

$\therefore P(x) =$

$(4x^2 + 2x)(2x^2 - x + 2) + 7x + 1$

**Synthetic Division**

This is a quick method for dividing polynomials **when the divisor is of the form $x - c$**

← 0x ☆

Divide: $2x^3 - 7x^2 + 5 \quad by \quad x - 3$

$x - \boxed{3}$ $c = 3$

☆ All terms of polynomial must be noted in division:

$$\begin{array}{r|rrrr} & x^3 & x^2 & x & c \\ 3 & 2 & -7 & 0 & 5 \\ & \downarrow & 6 & -3 & -9 \\ \hline x & 2 & -1 & -3 & \boxed{-4} \end{array}$$

coefficients

← Remainder

coefficients of polyn. 1 degree less

$f(x) = (x-3)(2x^2 - x - 3) - 4$

**The Remainder Theorem**

If the polynomial f(x) is divided by $x - c$, then the remainder is the value $f(c)$.

$x - \boxed{c}$ $f(c) = \underline{\text{remainder}}$

Find the remainder if $f(x) = x^3 - 4x^2 - 5$ use synthetic division and check using the remainder theorem.

1. divide by $x - 3$ $c = 3$
2. divide by $x + 2$ $c = -2$

$$\begin{array}{r|rrrr} & x^3 & x^2 & x & c \\ 3 & 1 & -4 & 0 & -5 \\ & \downarrow & 3 & -3 & -9 \\ \hline x & 1 & -1 & -3 & \boxed{-14} \end{array} \; R$$

$f(3) = 3^3 - 4(3^2) - 5$

$= 27 - 36 - 5$

$f(3) = \underline{-14}$

$$\begin{array}{r|rrrr} -2 & 1 & -4 & 0 & -5 \\ & & -2 & 12 & -24 \\ \hline & 1 & -6 & 12 & \boxed{-29} \end{array} \; R$$

$f(-2) = (-2)^3 - 4(-2)^2 - 5$

$= -8 - 16 - 5$

$f(-2) = \underline{-29}$

**Factor Theorem**

$$Let\ f\ be\ a\ polynomial\ function.\ Then\ (x-c)\ is\ a\ factor\ of\ f(x)$$
$$if\ and\ only\ if\ f(c)=0$$ $\Leftarrow$ remainder $= 0$!

Basically if $f(c)=0,\ then\ x-c\ is\ a\ factor\ of\ f(x)\ \ and\ \ c\ is\ a\ zero\ of\ f(x)$

Use the factor theorem to determine whether the function

$$f(x)=2x^3-x^2+2x-3$$

has the factor

a) $x-1$ $\Rightarrow c = 1$
b) $x+2$ $\Rightarrow c = -2$

$f(1) = 2(1^3) - 1^2 + 2(1) - 3$
$= 2 - 1 + 2 - 3$
$f(1) = 0$ $\therefore$ $x-1$ is a factor

$f(-2) = 2(-2^3) - (-2)^2 + 2(-2) - 3$
$= -16 - 4 - 4 - 3$
$f(-2) = -27$ $\therefore$ $x+2$ is NOT a factor

**Finding the roots**

Take a look at $P(x)=(x-2)(x-3)(x+4)$ multiplying the factors together we get:

$P(x)=x^3-x^2-14x+24$ Where did the constant 24 come from?

q = factors of 1 P = factors of 24

So, the constants of the factors multiplied out give us the constant of P(x). If the product of the zeros equals the constant then the zeros are all factors of the constant! We use this fact in the form of the **Rational Zeros Theorem**.

**Rational Zeros Theorem**

If the polynomial, P, has integer coefficients,

then every rational zero of $P$ is of the form $\dfrac{\boldsymbol{p}}{\boldsymbol{q}}$

**p** is a factor of the constant

**q** is a factor of the leading coefficient.

a. *So you need to find all the possible*
   $\pm p$ *values and* $\pm q$ *values to make all the* $\pm\frac{p}{q}$ *ratios*
   *One or more of the* $\pm\frac{p}{q}$ *ratios will be zeros of the polynomial.*
   *Determine zeros either by using the remainder theorem:* $P\left(\frac{p}{q}\right)=0$, *or if a calculator is allowed, use the graphing feature to locate the rational roots.*
b. *Once you find a zero, use synthetic division to reduce the polynomial into factors.*
c. *Keep following this process until you reach a quadratic factor then factor the quadratic, or use the Quadratic Formula to calculate last two factors.*

With graphing calculators we don't have to use the remainder theorem to find rational factors.. Remember! If you are not allowed to use a graphing calculator, you will want to plug into the polynomial each possible zero to see which one yields a remainder of zero, hence, a factor.

**Steps for finding the real zeros:**

1. List all possible $\frac{p}{q}$ ratios in reduced form.
2. Graph to see which ratios are zeros of the polynomial. [If no calculator is allowed, use the remainder theorem to make this determination.] expected to be able to work without calculator
3. The degree of the polynomial indicates the number of zeros of a function.
4. Use synthetic division to pare down the function to a quadratic factor and linear factors. **see below*
5. With the factors you can determine the zeros of the function.
6. If the quadratic factor is considered to be **irreducible or prime** if it cannot be factored over the real numbers.
7. If all real roots are expected then use the Quadratic formula on the irreducible quadratic to find the irrational roots (real) or the not so real imaginary roots. See 4.4.

* When the polynomial is divided by one of its binomial factors the resulting quotient is known as a **depressed polynomial.** Go figure! The depressed polynomial may be broken down in to other binomial factors if there are other rational roots.

1. List the possible rational zeros for the function below.
2. Then find the real zeros of the polynomial.
3. Factor completely.

$$f(x) = 2x^3 + 11x^2 - 7x - 6$$

$P \Rightarrow -6 \quad \pm\{1, 2, 3, 6\}$

$q \Rightarrow 2 \quad \pm\{1, 2\}$

Possible rational roots

$\pm\frac{P}{q} = \pm\{1, 2, 3, 6, \frac{1}{2}, \frac{3}{2}\}$

$f(1) = 2 + 11 - 7 - 6 = 0 \quad x = 1$

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 11 | -7 | -6 |
| | | 2 | 13 | 6 |
| | 2 | 13 | 6 | $\boxed{0}$ |

$ax^2 + bx + c$

$(x-1)(2x^2 + 13x + 6) = 0$

$a(c)$

$2(6)$

$12$

$1 \cdot 12$

$1 + 12 = 13$

$(2x^2 + x \mid + 12x + 6)$

$x(2x+1) + 6(2x+1)$

#3 $(x-1)(2x+1)(x+6) = f(x)$

#2 $x = 1, x = -\frac{1}{2}, x = -6$

Solve the equation:

$$3x^3 + 8x^2 - 7x - 12 = 0$$

$P \Rightarrow -12 \quad \pm\{1, 2, 3, 4, 6, 12\}$

$q \Rightarrow 3 \quad \pm\{1, 3\}$

Possible rational roots

$$\frac{p}{q} = \pm\left\{1, 2, 3, 4, 6, 12, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}\right\}$$

$x = -3, \ x = -1$

coeffs

$$\begin{array}{r|rrrrl}
-3 & 3 & 8 & -7 & -12 & x^3 \\
 & & -9 & 3 & 12 & \\
\hline
-1 & 3 & -1 & -4 & \boxed{0} & x^2 \\
 & & -3 & 4 & & \\
\hline
 & 3 & -4 & \boxed{0} & & x
\end{array}$$

$$f(x) = (x+3)(x+1)(3x-4) = 0$$

$$x = -3, \ x = -1, \ x = \frac{4}{3}$$

**Precalculus**
**Lesson: 4.3 Complex Zeros; Fundamental Theorem of Algebra**
**Mrs. Snow, Instructor**

Not all quadratic equations have real solutions. If we look at the complex number system, every quadratic equation has at least one complex solution; remember rational and irrational roots are in fact complex numbers. We just don't write them in the complex form. The fact that each polynomial function will have a complex solution brings about an important theorem.

**Fundamental Theorem of Algebra.**

> Every complex polynomial function $f(x)$ of degree $n \geq 1$ has at least one complex zero.

Another important theorem states that if we have the solution $a + bi$ then we must also have the solution $a - bi$.

Multiplying conjugates ?? $(a+b)(a-b) =$
(difference of 2 squares!) $= a^2 - b^2$

**Conjugate Pairs Theorem.**

> Let $f(x)$ be a polynomial function whose coefficients are real numbers. If $r = a + bi$ is a zero of $f$, the complex conjugate $\bar{r} = a - bi$ is also a zero of $f$.

Using the Conjugate Pairs Theorem

> A polynomial function $f$ of degree 5 whose coefficients are real numbers has the zeros 1, $5i$, and $1 + i$. Find the remaining two zeros.

$x = 5$ $\quad x = 5i$ $\quad x = 1 + i$

$x = -5i$ $\quad x = 1 - i$ conjugates

(a) Find a polynomial function $f$ of degree 4 whose coefficients are real numbers and that has the zeros 1, 1, and $-4 + i$.

(b) Graph the function found in part (a) to verify your result.

$x=1$ $\quad x=1 \quad x=-4+i \quad x=-4-i$

factors:

$(x-1) \quad (x-1) \quad (x+4-i) \quad (x+4+i)$

multiply out: conjugates

$(x-1)(x-1)\left(x+(4-i)\right)\left(x+(4+i)\right)$

$(x-1)(x-1)\left[x^2+x(4+i)+x(4-i)+(4-i)(4+i)\right]$

$(x^2-2x+1)\left(x^2+\cancel{4x}+\cancel{xi}+\cancel{4x}-\cancel{xi}+16-i^2\right)$

$16-(-1)$

$16+1=17$

$(x^2-2x+1)(x^2+8x+17)$

$$\begin{array}{rrrrr} x^4 & +8x^3 & +17x^2 & & \\ & -2x^3 & -16x^2 & -34x & \\ & & x^2 & +8x & +17 \end{array}$$

$f(x)=x^4+6x^3+2x^2-26x+17$

FYI:

$\sqrt{-1}=i$

Square each side

$-1=i^2$

Find the complex zeros of the polynomial function: 4th degree : 4 zeros

$$f(x) = 3x^4 + 5x^3 + 25x^2 + 45x - 18$$

Possible rat. roots

$p \quad \pm 1, 2, 3, 6, 9, 18$

$q \quad \pm 1, 3$

$\frac{p}{q} = \pm \left\{ 1, 2, 3, 6, 9, 18, \frac{1}{3}, \frac{2}{3} \right\}$

Calculator => $x = -2$

$$\begin{array}{r|rrrrr} -2 & 3 & 5 & 25 & 45 & -18 \\ & & -6 & 2 & -54 & 18 \\ \hline & 3 & -1 & 27 & -9 & \boxed{0} \end{array}$$

$(x+2)(3x^3 - x^2 + 27x - 9) = 0$ factor by grouping

$(x+2) \quad x^2(3x-1) + 9(3x-1)$

$(x+2)(x^2+9)(3x-1) = 0$

$x+2=0 \qquad x^2+9=0 \qquad 3x-1=0$

$x=-2 \qquad x^2=-9 \qquad 3x=1$

$x = \pm\sqrt{3}i \qquad x = \frac{1}{3}$