Precalculus
Lesson 4.4: Properties of Rational Functions
Mrs. Snow, Instructor

When dealing with ratios of integers, they are identified as rational numbers. When we look at ratios of polynomials, we call them **rational functions.**

A **rational function** is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions and q is not the zero polynomial. The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

Find the domain of the rational functions:

$R(x) = \dfrac{2x^3 - 4}{x + 5}$	$R(x) = \dfrac{1}{x^2 - 4}$
$R(x) = \dfrac{x^3}{x^2 + 1}$	$R(x) = \dfrac{x^2 - 1}{x - 1}$

Graph and analyze. What happens at $x = 0$? As x gets closer to 0? What happens as $x \to \infty$?

$$R(x) = \frac{1}{x} \qquad\qquad H(x) = \frac{1}{x^2}$$

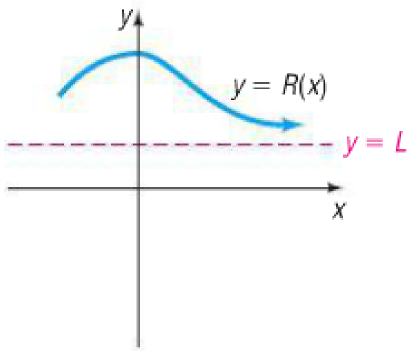
Graph the rational function using transformations:

$$R(x) = \frac{1}{(x-2)^2} + 1$$

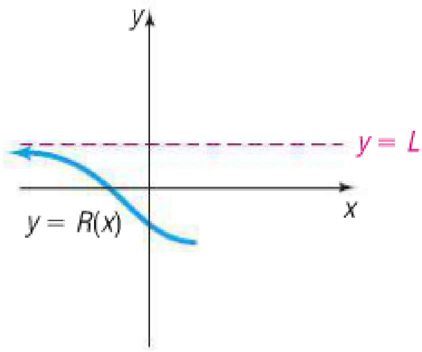
Asymptotes (pg. 219)

Let R denote a function:
If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R.
If, as x approaches some number c, the values $|R(x)| \to \infty$, then the line $x = c$ is a **vertical asymptote** of the graph of R. The graph of R never intersects a vertical asymptote.

Horizontal asymptotes:

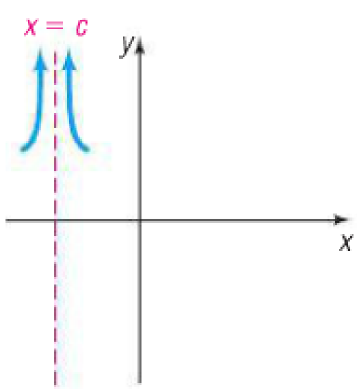


(a) End behavior: As $x \to \infty$, the values of $R(x)$ approach L [symbolized by $\lim_{x \to \infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

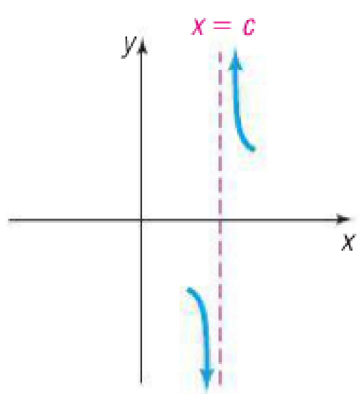


(b) End behavior: As $x \to -\infty$, the values of $R(x)$ approach L [symbolized by $\lim_{x \to -\infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

Vertical asymptotes:



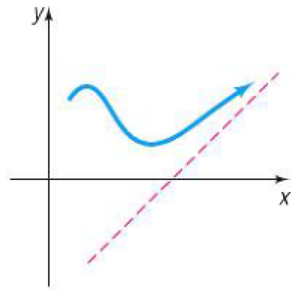
(c) As x approaches c, the values of $R(x) \to \infty$ [for $x < c$, this is symbolized by $\lim_{x \to c^-} R(x) = \infty$; for $x > c$, this is symbolized by $\lim_{x \to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.



(d) As x approaches c, the values of $|R(x)| \to \infty$ [for $x < c$, this is symbolized by $\lim_{x \to c^-} R(x) = -\infty$; for $x > c$, this is symbolized by $\lim_{x \to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

NOTE: Horizontal asymptotes may be intersected by the graph of a function! *<u>The graph of a function will never intersect a vertical asymptote.</u>*

There is also another type of asymptote, **OBLIQUE ASYMPTOTE**.



Vertical Asymptotes

- Factor denominator and set it equal to zero. The values where the denominator goes to zero will be the vertical asymptotes; these are the domain restrictions and will graphically be seen as vertical asymptote(s).

Fid the vertical asymptotes, if any, of the graph of each rational function.

(a) $F(x) = \dfrac{x+3}{x-1}$

(b) $R(x) = \dfrac{x}{x^2-4}$

(c) $H(x) = \dfrac{x^2}{x^2+1}$

(d) $G(x) = \dfrac{x^2-9}{x^2+4x-21}$

Horizontal and Oblique Asymptotes

Last year we learned the little mnemonic of BOBO BOTN EATSDC… This is great, except there was one thing we did not discuss in Alg II and that was oblique asymptotes. So a new set of rules comes into play.

If the ratio for the rational function is **proper** (fraction); the degree of the numerator is less than the degree of the denominator, then it is a BOBO situation and the horizontal asymptote is $y = 0$.

Find the horizontal if one exists

Find the horizontal asymptote, if one exists, of the graph of

$$R(x) = \frac{x - 12}{4x^2 + x + 1}$$

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + \cdots a_1 x + a_0}{b_m x^m + \cdots b_1 x + b_0}$$

Numerator has a degree of ***n*** and denominator has a degree of ***m***

If the ratio for the rational function is **improper;** the degree of the numerator is greater than or equal to the degree of the denominator: $n \geq m$. **Divide using long division:**
This will yield the following:

$$R(x) = \frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)}$$

1. Degrees of numerator and denominator are equal $n = m$:
   This is what we described last year as EATSDC; the horizontal is described by a line equal to the ratio of the leading coefficients.
2. Degree of numerator is one more than the denominator $n = m + 1$:
   Divide, the quotient obtained is of the form of $ax + b$, the line $y = ax + b$ is the oblique asymptote.
3. Degree of the numerator is two more than the denominator $n = m + 2$:
   Divide, the quotient is a polynomial for degree 2 or higher. *R* will have neither horizontal nor obliqui asymptotes. The graph for large values of $|x|$ will behave like the graph of the quotient.

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1}$$

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1}$$

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1}$$