Lesson 9-1
Inverse Variation

When the ratio of two variables has a constant (unchanged) ratio, their relationship is called a direct variation. We say that \( y \) varies directly as \( x \). The constant ratio, \( k \), is called the constant of variation.

\[
\frac{y}{x} = k, \quad \text{or} \quad y = kx
\]

Note: in a linear situation the constant of variation is our slope:

\[
\frac{\text{rise}}{\text{run}} = \frac{y}{x}
\]

In direct variation problems, we will see that as one variable increases the other increases. Likewise as one decreases so will the other decrease.

Melissa’s weekly salary, \( s \), varies directly as the number of hours, \( h \), that she works. Write an equation that describes this relation. Solve for the constant of variation.

Melissa’s check shows she worked 32 hours and it is the amount of $363.20. What is her hourly rate?

\[
s = kh \quad \quad k = \frac{363.20}{32 \ \text{hr}} = \$11.35/\text{hr}
\]

According to Hooke's Law, the force needed to stretch a spring is proportional to the amount the spring is stretched. If fifty pounds of force stretches a spring five inches, how much will the spring be stretched by a force of 120 pounds?

\[
F = kd \quad \quad k = \frac{50}{5} = 10 \quad \quad 120 = 10d \quad \quad \frac{120}{10} = 12 \text{ inches}
\]

If \( y \) varies directly as \( x^2 \) and \( y = 8 \) when \( x = 2 \), find \( y \) when \( x = 1 \). Write the equation of variation.

\[
y = kx^2 \quad \quad y = 2x^2 \quad \quad y = 2(1^2) \quad \quad y = 2
\]

\[
8 = k(4) \quad \quad \frac{8}{4} = 2
\]
The opposite of direct variation is known as **Inverse Variation**. In an inverse variation, the values of the two variables change in an opposite manner, that is, as one value increases, the other decreases.

\[ xy = k, \quad \text{so} \quad y = \frac{k}{x}, \quad \text{or} \quad x = \frac{k}{y} \]

Let's look at an example. How long will it take a cyclist to bike 8 miles? Well that depends on his speed. A biker traveling at 8 mph can cover 8 miles in 1 hour. If the biker's speed decreases to 4 mph, it will take the biker 2 hours to cover the same distance.

<table>
<thead>
<tr>
<th>Rate (mi/hr)</th>
<th>Time (hr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
</tr>
</tbody>
</table>

Notice that as the rate decreases, the time increases. Cut the rate in half, the time doubles. Our rate equation may be written as \( t = \frac{d}{r} \), at distance equal to 8 miles we get: \( t = \frac{8}{r} \).

Graphically, we see can see the relation between time and rate (speed).

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Given \( y \) varies inversely as \( x \). Write a variation function when \( y = 1.4 \) and \( x = 0.3 \)

So: \( k =? \) and our inverse variation function is:

\[ y = \frac{1.4}{x} \]

or

\[ y = \frac{\text{42}}{x} \]

\( 1.4 \cdot 0.3 = 1 \cdot 0.42 \)
Determine if the relationship between the values is direct variation, inverse variation or neither. Write an equation if possible.

\[
\begin{array}{c|c|c}
\hline
x & 2 & 4 \\
\hline
y & 3.2 & 1.6 \\
\hline
\end{array}
\quad \begin{array}{c|c|c}
\hline
x & 0.8 & 0.6 \\
\hline
y & 0.9 & 1.2 \\
\hline
\end{array}
\]

\( \frac{y}{x} = k \quad xy = k \)

\[
\begin{array}{c|c|c}
\hline
x & 2 & 5 & 8 & 9.5 \\
\hline
y & 14 & 35 & 56 & 66.5 \\
\hline
\end{array}
\quad \begin{array}{c|c|c|c}
\hline
x & 2 & 2.5 & 5 & 6 \\
\hline
y & 30 & 24 & 12 & 10 \\
\hline
\end{array}
\]

\( k = xy \quad 2 \quad 7.2 \quad 7.2 \quad \text{constant} \)

\( y = \frac{7.2}{x} \)

\( y = \frac{60}{x} \)

Write the function that models each inverse variation. Then find \( y \) when \( x = 9 \).

\[ x = 3 \text{ when } y = -5 \]

\[
\begin{align*}
xy &= k \\
3(-5) &= k = -15 \\
y &= -\frac{15}{x} \\
\frac{y}{x} &= 1.67
\end{align*}
\]

The inverse variation contains the ordered pair: \((6, 3)\)

\[
\begin{align*}
xy &= k = (6)(3) = 18 \\
y &= \frac{18}{x} \\
\frac{y}{x} &= 3 \\
y &= 3
\end{align*}
\]

**Combined Variation**

Variation is not only for linear relationships. We can just as easily have a situation where \( y \) varies inversely with \( x^2 \), such that: \( y = \frac{k}{x^2} \). Also, we can have situations where we have what is called a joint variation. Here a variable will vary jointly with two other variables: \( z = kxy \). Let's put some of these combinations into table form:

<table>
<thead>
<tr>
<th>Combined Variation</th>
<th>Equation form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z ) varies jointly with ( x ) and ( y ) ( (\text{direct} \times \text{with} \text{both} \text{keys}) )</td>
<td>( z = kxy )</td>
</tr>
<tr>
<td>( z ) varies jointly with ( x ) and ( y ) and inversely with ( w ) ( (\text{two} \text{in}, \text{\text{denominator}}) )</td>
<td>( z = \frac{kxy}{w} )</td>
</tr>
<tr>
<td>( z ) varies directly with ( x ) and inversely with the product ( wy )</td>
<td>( z = \frac{kx}{wy} )</td>
</tr>
</tbody>
</table>
Given that \( z \) varies directly with \( x \) and inversely with \( y \). Write a variation function when \( x = 6, y = 2, \) and \( z = 15 \)

\[
\frac{z}{y} = k \quad \Rightarrow \quad 15 = \frac{k \cdot 6}{2} \quad \Rightarrow \quad z = \frac{5x}{y}
\]

\[
15 = 3k \quad \frac{15}{3} = k = 5
\]

Given that \( z \) varies jointly with \( x \) and \( y \). Write a variation function when \( x = 2, y = 3 \) and \( z = 60 \)

\[
z = k \cdot x \cdot y \quad 60 = k \cdot (2)(3)
\]

\[
60 = 6k \quad \frac{60}{6} = k = 10
\]

Describe the combined variation that is modeled by each formula:

\[ a) \quad A = \frac{k}{r^2} \quad \text{Area varies directly with the square of the radius} \]

\[ b) \quad h = \frac{2A}{b} \quad \text{Height varies directly with the area and inversely with the base} \]

The volume \( V \) of a tetrahedron varies jointly with its altitude \( h \) and base of area \( B \). Find the formula that models this joint variation. Given that the tetrahedron has an altitude of 5 cm., a base area of 6 cm\(^2\), and a volume of 10 cm\(^3\)

\[
V = k \cdot h \cdot B \quad 10 = k \cdot (5)(6) \quad \Rightarrow \quad V = \frac{1}{3} h \cdot B
\]

\[
10 = 30k \quad \frac{10}{30} = \frac{1}{3} = k
\]
Other stuff

Given a direct variation, find the missing variable for the pair of values: \((4,6), (x, 3)\)

\[ y = kx \]
\[ \frac{y}{x} = k = \frac{6}{4} = \frac{3}{2} \]
\[ \frac{2}{x} = \frac{3}{2} \cdot \frac{x}{2} \]
\[ \frac{2}{x} = x \]

Given an inverse variation, find the missing variable for the pair of values: \((4,6), (x, 3)\)

\[ \frac{y}{x} = k \]
\[ 4(6) = k = 24 \]
\[ 3 = \frac{24}{x} \]
\[ x = \frac{24}{3} = 8 = x \]