Algebra II
Lesson 6.5/6.6 Theorems about Polynomial Functions
Part I
Mrs. Snow, Instructor

The sections of 6.5 and 6.6 are being combined into a single topic with two parts. Part I will cover theorems that will help us to determine all possible roots when only some of the roots are known.

**Irrational Root Theorem:** Given a polynomial with rational coefficients and \( \sqrt{b} \) is irrational. If \( a + \sqrt{b} \) is a root, then you will also have the root of \( a - \sqrt{b} \); these are called *conjugates*

| Opposite sign on 2nd turn |

**Imaginary Root Theorem:** If the imaginary number of \( a + bi \) is a root of a polynomial with real coefficients, then the conjugate, \( a - bi \) is also a root. Again note these are *conjugates*. These roots are called complex conjugates

To find these roots, you will reduce the polynomial down to linear factors and a quadratic factor by dividing the real factors into the original polynomial. When you get a quadratic factor, you will use the Quadratic Formula to solve.

A polynomial equation has the following roots, find two additional roots.

<table>
<thead>
<tr>
<th>2 - ( \sqrt{7} ) and 3 + 2( \sqrt{6} )</th>
<th>1 + ( \sqrt{3} ), and -( \sqrt{11} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2} )</td>
<td>1 - ( \sqrt{3} ), ( +\sqrt{11} )</td>
</tr>
<tr>
<td>2 + ( \sqrt{3} ), 3 - 2( \sqrt{6} )</td>
<td>( 1 - \sqrt{3} ), ( +\sqrt{11} )</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>3 - ( i ), and 2( i )</th>
<th>12 + 3( i ), and 4 - ( i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 + ( i ), -2( i )</td>
<td>( 12 - 3i ), 4 + ( i )</td>
</tr>
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</table>

Note - in the Quadratic Formula the conjugates are "built in" with the \( \pm \) conjugates

\[
\chi = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

Possible complex/irrational value
Find a 3\textsuperscript{rd} degree polynomial equation with rational coefficients that has the given roots:

\[ 1 \text{ and } 3i \Rightarrow -3i \]
\[ x = 1 \quad x = 3i \quad x = -3i \]
\[ x - 1 = 0 \quad x - 3i = 0 \quad x + 3i = 0 \]
\[ (x - 1)(x - 3i)(x + 3i) = 0 \]
\[ (x - 1)(x^2 - 9) = 0 \]
\[ (x - 1)(x^2 + 9) = 0 \]
\[ x^3 + 9x \]
\[ -x^2 - 9 \]
\[ x^3 - x^2 + 9x - 9 = 0 \]

\[ 2 + i \text{ and } -3 \]
\[ x = 2 + i \quad x = 2 - i \quad x = -3 \]
\[ x + 3 = 0 \quad x - 2 - i = 0 \quad x - 2 + i = 0 \]
\[ (x + 3)(x - 2 - i)(x - 2 + i) = 0 \]
\[ (x + 3)(x - 2)(x - 2) = 0 \]
\[ (x + 3)(x - 2)(x^2 - 4x + 4 + 1) = 0 \]
\[ x^3 - 4x^2 + 9x \]
\[ 3x^2 + 15 \]
\[ x^3 - x^2 + 5x + 15 = 0 \]

Find a 4\textsuperscript{th} degree polynomial equation with rational coefficients that has the given roots:

\[ 5 - i \text{ and } \sqrt{2} \Rightarrow \text{conjugates } 5 + i, \quad -\sqrt{2} \]
\[ x = 5 - i \quad x = 5 + i \quad x = \sqrt{2} \quad x = -\sqrt{2} \]
\[ x - 5 + i = 0 \quad x - 5 - i = 0 \quad x - \sqrt{2} = 0 \quad x + \sqrt{2} = 0 \]
\[ (x - 5 + i)(x - 5 - i)(x - \sqrt{2})(x + \sqrt{2}) = 0 \]
\[ (x - 5)^2 - (2i)^2 \quad (x^2 - 2) = 0 \]
\[ (x^2 - 10x + 25 + 4)(x^2 - 2) = 0 \]
\[ (x^2 - 2)(x^2 - 10x + 29) = 0 \]
\[ x^4 - 10x^3 + 26x^2 \]
\[ -2x^2 + 20x - 52 = 0 \]
\[ x^4 - 10x^3 + 24x^2 + 20x - 52 = 0 \]
Fundamental theorem of Algebra

In 1797 Carl Gauss proved what is known as the Fundamental Theorem of Algebra. It states that a polynomial of degree \( n \geq 1 \) has at least one complex zero. In essence, the fundamental theorem of algebra guarantees that every polynomial has a complete factorization, if we are allowed to use complex numbers \((a + bi)\). Remember that a real number may be written as a complex number.

**Fundamental Theorem of Algebra; translation:** An nth degree polynomial equation has exactly \( n \) roots; at least one of them will be complex.

You can often find all the zeros of a polynomial function by using a combination of graphing, Factor Theorem, polynomial division, the Remainder Theorem and the Quadratic Formula.

Find the number of complex roots, and the possible number of real roots

\[
x^5 + x^3 + 2x^2 - 6 = 0
\]

- \( x^5 \) degree = 5 complex roots
- Imaginary roots will be paired \( \pm i \)
- \((a+) (a-) (c+) (c-) (d)\)
- At most 4 imaginary \( \rightarrow 1 \) real
- 2 imaginary \( \rightarrow 3 \) real
- 0 " " \( \rightarrow 5 \) real

\[
x^{10} + x^8 + x^4 + 3x^2 - x + 1 = 0
\]

- 10 complex roots (degree)
- Imaginary roots pair up \( \pm i \)
- \((a+) (a-) (b+)(b-) (c+) (c-) (d+)(d-) (e+)(e-)\)
- So if all imaginary \((10) - 10 \) real roots
- 8 " " \( \rightarrow 2 \) real
- 6 " " \( \rightarrow 4 \) real
- 4 " " \( \rightarrow 6 \) real
- 2 " " \( \rightarrow 8 \) real
- 0 " " \( \rightarrow \text{all (10) real} \)