## Algebra 2

## Lesson 4-5/4-5: Inverses: $2 \times 2$ and $3 \times 3$

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Recall the number 1 is the multiplicative identity for any real number $\boldsymbol{a}$. That is: $a \cdot 1=a$, in other words, the product of a number and the multiplicative identity is the number. Well, our square matrices also have multiplicative identities too. The matrix identity is called, the multiplicative identity matrix; it is equivalent to " 1 " in matrix terminology. So, a matrix multiplied by I is equal to the matrix.

The identity matrix of a $2 \times 2$ and a $3 \times 3$ square matrix are:

$$
I_{2}=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] \quad \text { and } \quad I_{3}=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

Note: the identity matrix is Identified with a capital I and a subscript indicating the dimensions; it consists of a diagonal of ones and the corners are filled in with zeros. Its dimensions are square.

Example: Multiply A by the identity matrix

$$
\left[\begin{array}{cc}
3 & 4 \\
-2 & 2
\end{array}\right] \times\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]=\left[\begin{array}{cc}
3 \cdot 1+4 \cdot 0 & 3 \cdot 0+4 \cdot 1 \\
-2 \cdot 1+2 \cdot 0 & -2 \cdot 0+2 \cdot 1
\end{array}\right]=\left[\begin{array}{cc}
3 & 4 \\
-2 & 2
\end{array}\right]
$$

Inverses: A number times its inverse (A.K.A. reciprocal) is equal to 1 so is a matrix times its inverse equal to "1." When two matrices are multiplied, and the product is the identity matrix, we say the two matrices are inverses.

Now we can set about to find inverses and verify if a matrix is an inverse of another.
It is a process, a pattern to follow and not that bad.
Example: Is $\mathbf{B}$ the inverse of $\mathbf{A}$ ?:
\(\left.$$
\begin{array}{|c|l|}\hline A=\left[\begin{array}{ll}2 & 3 \\
1 & 2\end{array}\right] & \text { and } B=\left[\begin{array}{cc}2 & -3 \\
-1 & 2\end{array}\right]\end{array}
$$ \begin{array}{l}If B is the inverse then A B should <br>

equal the identity matrix, does it?\end{array}\right]\)| $\left[\begin{array}{ll}2 & 3 \\ 1 & 2\end{array}\right]\left[\begin{array}{cc}2 & -3 \\ -1 & 2\end{array}\right]=\left[\begin{array}{cc}2 \cdot 2+3 \cdot-1 & 2 \cdot-3+3 \cdot 2 \\ 1 \cdot 2+2 \cdot-1 & 1 \cdot-3+2 \cdot 2\end{array}\right]=\left[\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right]$ | Yes, $B$ is the inverse of $A$ |
| :--- | :--- |

The notation for the inverse of a matrix is the matrix letter identity and a-1 superscript, that is: $\mathbf{A}^{\mathbf{- 1}}$. In other words:

$$
\boldsymbol{A} \boldsymbol{A}^{-1}=\boldsymbol{A}^{-1} \boldsymbol{A}=\boldsymbol{I} \quad \text { (Remember I, identity matrix, is the " } 1 \text { " for matrices) }
$$

Well, how do we find the inverse? We do scalar multiplication with the value of $\frac{\mathbf{1}}{\operatorname{det} A}$ times a mixed-up version of matrix A!! Look below to see what I mean, it is not that bad.....

$$
\text { For matrix } \mathrm{A}=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \text { the inverse is: } \quad \mathrm{A}^{-1}=\frac{1}{\operatorname{det} A}\left[\begin{array}{cc}
d & -b \\
-c & a
\end{array}\right]
$$

Note: $\operatorname{det} \mathrm{A}$ in the denominator. Determinants cannot be equal to 0 as $\frac{\mathbf{1}}{\boldsymbol{\operatorname { d e t }} \boldsymbol{A}}$ would be undefined. So, a matrix with a determinant of 0 has no inverse and is called a singular matrix.
Example: Find the inverse of the matrix,

$$
\begin{gathered}
A=\left[\begin{array}{cc}
-2 & 2 \\
3 & -4
\end{array}\right] \\
\operatorname{det} A=8-6=2 \\
A^{-1}=\frac{1}{2}\left[\begin{array}{ll}
-4 & -2 \\
-3 & -2
\end{array}\right]=\left[\begin{array}{cc}
-2 & -1 \\
-3 / 2 & -1
\end{array}\right]
\end{gathered}
$$

1. verify that $\operatorname{det} A \neq 0$
2. set up inverse equation (note: switch a and d, and make $c$ and b opposite sign)

Well, what good is an inverse matrix? Yesterday we discussed how inverse matrices are used in data encryption. That application involves algebra and a process that uses the inverse of a matrix to solve a system of equations. This process is in fact quite similar to solving an equation like $5 x=20$. To solve for $x$ we multiply each side by $1 / 5$, the inverse of 5 .

To solve a matrix equation, $\boldsymbol{A} \boldsymbol{X}=\boldsymbol{B}$ for $X$, you simply multiply each side by $A^{-1}$

1. $A^{-1} A X=A^{-1} B$, so we are left with X on the left side which is what we are solving for. So: $X=A^{-1} B$
2. WARNING!!! You must always keep the order of the matrices uniform! $A^{-1} B$ is NOT the same as $B A^{-1}$ !

Example: Solve the matrix equation for $X$

| $\begin{gathered} {\left[\begin{array}{cc} -2 & -5 \\ 1 & 3 \end{array}\right] X=\left[\begin{array}{c} -2 \\ 2 \end{array}\right]} \\ A^{-1}\left[\begin{array}{cc} -2 & -5 \\ 1 & 3 \end{array}\right] X=A^{-1}\left[\begin{array}{c} -2 \\ 2 \end{array}\right] \\ X=A^{-\mathbf{1}}\left[\begin{array}{c} -\mathbf{2} \\ \mathbf{2} \end{array}\right] \end{gathered}$ | 1. Multiply each side by the inverse. <br> 2. Multiplication by the inverse leaves $X$ on the left side. Simplify the right side |
| :---: | :---: |
| $\operatorname{det} A=\left\|\begin{array}{cc} -2 & -5 \\ 1 & 3 \end{array}\right\|=-2 \cdot 3-(-5) 1=-6-(-5)=-1$ | 3. Calculate det A. |
| $A^{-1}=\frac{1}{\operatorname{det} A}\left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right]=\frac{1}{a d-b c} \cdot\left[\begin{array}{cc} 3 & 5 \\ -1 & -2 \end{array}\right]=\quad-1\left[\begin{array}{cc} 3 & 5 \\ -1 & -2 \end{array}\right]=\left[\begin{array}{cc} -3 & -5 \\ 1 & 2 \end{array}\right]$ | 4. Using the determinant find $A^{-1}$ |
| $X=A^{-1} B \quad=\left[\begin{array}{cc} -3 & -5 \\ 1 & 2 \end{array}\right]\left[\begin{array}{c} -2 \\ 2 \end{array}\right]=\left[\begin{array}{cc} -3(-2)+(-5)(2) \\ (1)(-2) & (2)(2) \end{array}\right]=\left[\begin{array}{c} -4 \\ 2 \end{array}\right]=X$ | 5. With $A^{-1}$ simplify the right side of the equation to solve for $X$. |

As with $2 \times 2$ matrices, when we multiply a $3 \times 3$ matrix with its inverse, we will get the identity matrix, $I_{3}$. So we can also show that a $3 \times 3$ matrix is the inverse of another $3 \times 3$ matrix in the same fashion as our $2 \times 2$ example.

The methods for finding $2 \times 2$ inverses also holds true for $3 \times 3$ inverses, BUT we use a calculator!!!

1. Using a calculator, enter the data for a $3 x 3$ matrix and the matrix located on the right side of the equal sign
2. Now to calculate the inverse hit $\mathbf{2}^{\text {nd }}$ MATRIX select the matrix you want the inverse for and hit ENTER
3. Hit $\mathbf{x}^{-1}$ (for example: $[A]^{-1}$ ) ENTER the view screen will show the inverse of the $3 \times 3$ matrix.
4. With the matrix inverse on the screen hit * (times) $\mathbf{2}^{\text {nd }}$ Matrix [B] ENTER (will show Ans *[B], that is our inverse times the $B$ matrix). The resulting matrix will be our answer, the matrix that equals $X$.
Below shows how matrix equations may be solved by using the inverse.

Example: Solve the matrix equation:

$$
\begin{gathered}
A^{-1} A X=A^{-1} B \\
X=A^{-1} B
\end{gathered}
$$

Find $\mathrm{A}^{-1}$ using the calculator
solve for $X$

$$
\left[\begin{array}{ccc}
0 & 0 & 2 \\
1 & 3 & -2 \\
1 & -2 & 1
\end{array}\right] \quad X=\left[\begin{array}{c}
6 \\
-11 \\
8
\end{array}\right]
$$

$$
\left[\begin{array}{ccc}
.1 & .4 & .6 \\
.3 & .2 & -.2 \\
.5 & 0 & 0
\end{array}\right]=A^{-1}
$$

$$
X=\left[\begin{array}{ccc}
.1 & .4 & .6 \\
.3 & .2 & -.2 \\
.5 & 0 & 0
\end{array}\right] \times\left[\begin{array}{c}
6 \\
-11 \\
8
\end{array}\right]=\left[\begin{array}{c}
1 \\
-2 \\
3
\end{array}\right]
$$

$$
A^{-1} \times B=X
$$

To keep your thoughts straight, see what you are doing, and what needs to be done next YOU MUST SHOW YOUR STEPS!
PRACTICE

1. Solve:
$\left[\begin{array}{ll}7 & 5 \\ 3 & 2\end{array}\right] X=\left[\begin{array}{l}-9 \\ -4\end{array}\right]$
2. Solve: $\left[\begin{array}{ccc}0 & 0 & 2 \\ 1 & 3 & -2 \\ 1 & -2 & 1\end{array}\right] \quad X=\left[\begin{array}{c}0 \\ -6 \\ 19\end{array}\right]$
3. Determine if the matrix has an inverse:
$\left[\begin{array}{ll}1 & 4 \\ 1 & 3\end{array}\right]$
4. Are the matrices multiplicative inverses?
$\left[\begin{array}{cc}2 & .5 \\ 5 & 1\end{array}\right], \quad\left[\begin{array}{cc}-2 & 1 \\ 10 & -4\end{array}\right]$
